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Immersed Spheres in 4-Manifolds and The Immersed Thom Conjecture

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1. Introduction

The introduction of the Seiberg - Witten monopole equations ([SW1],[SW2],[W]) has served to make the study of smooth 4-manifolds more accessible. Many of the important earlier theorems regarding Donaldson invariants of smooth 4-manifolds have more easily proven analogues in Seiberg-Witten theory [J]. Further, new fundamental results have quickly appeared, most notably the proof of the Thom conjecture [KM], the verification of the nontriviality of the Seiberg-Witten invariants for symplectic manifolds [T1], and the results of C. Taubes which relate Seiberg-Witten invariants with the theory of pseudo-holomorphic curves and certain Gromov invariants for symplectic manifolds ([T2],[T3]).

A few days after an MIT lecture in which Witten introduced these monopole equations to the mathematical public, Cliff Taubes visited Cal Tech and UC Irvine and described to us these equations. Shortly thereafter, Tom Mrowka travelled to UC Berkeley to connect with Peter Kronheimer. They quickly worked out a Weitzenbock argument to show that for minimal surfaces of general type the canonical class is a diffeomorphism invariant and that the Seiberg-Witten invariant vanished for manifolds with positive scalar curvature. Although this had already been realized by Witten [W], this got us all excited – for this was a result that we had expected but could not yet prove using Donaldson theory.

Both of us had just completed a series of results concerning Donaldson invariants in which we studied the effect of *embedded* spheres on the Donaldson invariants. In effect, we showed that understanding cut and paste arguments for 4-manifolds which are split along ordinary lens spaces provided insights into the structure of the Donaldson invariants ([FS1],[FS2], [FS3]). Cliff Taubes called the second author and pointed out that lens spaces have positive scalar curvature and that all the gluing arguments necessary for our work were much more trivial in this newer theory. Initially, the three of us, and independently Kronheimer and Mrowka, quickly recast all of our earlier work in this newer setting. Most of this will appear in [J].

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The first part of this paper is partly historical and reports on one of the early successes — a proof using the Seiberg-Witten invariants of the immersed Thom conjecture for the projective plane. Recall that the classical Thom Conjecture states that the genus of a smoothly embedded surface F in $\mathbf{CP}^2$ representing d times the generator H of $H_2(\mathbf{CP}^2;\mathbf{Z})$ must be at least the genus of a nonsingular complex curve of degree d. I.e.

$$g(F) \geq \frac{(d-1)(d-2)}{2}.$$

Instead of representing 2-dimensional homology classes by embedded surfaces, one can represent them by immersed 2-spheres. This leads to what we call the Immersed Thom Conjecture. In this paper we shall discuss a proof of following theorem.

Theorem 1.1 (The Immersed Thom Conjecture). *Suppose that a 2-sphere S is immersed in $\mathbf{CP}^2$ with p positive double points, and suppose that its image represents $dH \in H_2(\mathbf{CP}^2;\mathbf{Z})$. Then*

$$p \geq \frac{(d-1)(d-2)}{2}.$$

Within a few days of our discovery of a proof of this theorem, Kronheimer and Mrowka announced their proof of the full Thom conjecture [KM] which implies the immersed version; first remove all the positive double points of the immersion by adding handles to increase the genus by exactly p and then blow up to remove the negative double points and follow the proof of [KM]. However, our proof also gives new information about representing homology classes in the rational surface $\mathbf{CP}^2\#q\overline{\mathbf{CP}}^2$ by immersed spheres.

Theorem 1.2. *Let $\alpha = dH + \sum_1^q a_iE_i \in H_2(\mathbf{CP}^2\#q\overline{\mathbf{CP}}^2;\mathbf{Z})$. Then if $d \geq 2$ and*

$$d^2 - 3d \geq \sum_1^q (a_i^2 - a_i) + 2p$$

the class α cannot be represented by an immersed 2-sphere with p positive double points.

The question of representability by smoothly embedded 2-spheres ($p = 0$) has been well-studied in the literature in the case where $\alpha \cdot \alpha \geq 0$ where one may apply Donaldson's theorems about the realization of intersection forms. See [L] for a survey of results. The chief interest of the above theorem is where $\alpha \cdot \alpha < 0$. This is not implied by [KM].

Tom Mrowka also had an early proof of Theorem 1.1.

The purpose of the second part of this paper is to apply the proof of Theorem 1.2 together with the blowup formula for Seiberg-Witten invariants (Theorem 1.4 below whose proof we outline in §4) to manifolds with $b^+ > 1$ to obtain a very general adjunction formula for immersed spheres:

Theorem 1.3 (Generalized Adjunction Formula for Immersed Spheres). *Suppose that X is an arbitrary smooth 4-manifold with $b^+ > 1$ and that L is a characteristic line bundle with $SW_X(L) \neq 0$ and $\dim M_X(L) = \sum_{i=1}^r \ell_i(\ell_i + 1)$ with each integer $\ell_i \geq 0$. If $x \neq 0 \in H^2(X;\mathbf{Z})$ is represented by an immersed sphere with p positive double points, then either*

$$2p - 2 \geq x^2 + |x \cdot L| + 4\sum_{i=1}^{r} \ell_i, \quad p \geq r$$

$$2p - 2 \geq x^2 + |x \cdot L| + 4\sum_{i=1}^{p} \ell_i + 2\sum_{i=p+1}^{r} \ell_i, \quad p < r$$

or

$$SW_X(L) = \begin{cases} SW_X(L + 2x), & \text{if } x \cdot L \geq 0 \\ SW_X(L - 2x), & \text{if } x \cdot L \leq 0. \end{cases}$$

Here $SW_X(L)$ is the Seiberg-Witten invariant for L and $\dim M_X(L) = \frac{1}{4}(c_1(L)^2 - (3\,\mathrm{sign} + 2\,\mathrm{e})(X))$ is the formal dimension of the moduli space $M_X(L)$ of solutions to the Seiberg-Witten monopole equations. Note that in general there are several ways to write any even number as a sum of triangular numbers $\ell(\ell + 1)$. Theorem 1.3 is optimized by letting r be as large as possible.

Theorem 1.4 (Blowup Formula). *Suppose that X is an arbitrary smooth 4-manifold with $b^+ > 1$. If $\dim M_X(L) - r(r+1) \geq 0$, then*

$$SW_X(L) = SW_{X\#\overline{\mathbf{CP}}^2}(L \pm (2r+1)E).$$

Here we confuse line bundles with their first Chern classes and write the line bundles additively. Also $E \in H^2(\overline{\mathbf{CP}}^2;\mathbf{Z})$ is the Poincaré dual of the exceptional divisor of $\overline{\mathbf{CP}}^2$. Theorems 1.3 and 1.4 were discovered before the proof of Theorem 1.1.

2. Seiberg-Witten Invariants

Suppose we are given a spin^c structure on an oriented closed Riemannian 4-manifold X. Let W^+ and W^- be the associated spin^c bundles with $L = \det W^+ = \det W^-$ the associated determinant line bundle. Since $c_1(L) \in H^2(X;\mathbf{Z})$ is a characteristic cohomology class, i.e. has mod 2 reduction equal to $w_2(X) \in H^2(X;\mathbf{Z}_2)$, we refer to L as a characteristic line bundle. We will confuse a characteristic line bundle L with its first Chern class $L \in H^2(X;\mathbf{Z})$. For simplicity we assume that $H^2(M;\mathbf{Z})$ has no 2-torsion so that the set $Spin^c(X)$ of spin^c structures on X is precisely the set of characteristic line bundles on X.

Clifford multiplication, c, maps T^*X into the skew adjoint endomorphisms of $W^+ \oplus W^-$ and is determined by the requirement that $c(v)^2$ is multiplication by $-|v|^2$. Thus c induces a map

$$c : T^*X \to \mathrm{Hom}(W^+, W^-).$$

The 2-forms $\Lambda^2 = \Lambda^+ \oplus \Lambda^-$ of X then act on W^+ leading to a map $\rho : \Lambda^+ \to \mathrm{su}(W^+)$. A connection A on L together with the Levi-Civita connection on the tangent bundle of X induces a connection $\nabla_A : \Gamma(W^+) \to \Gamma(T^*X \otimes W^+)$ on W^+. This connection, followed by Clifford multiplication, induces the Dirac operator $D_A : \Gamma(W^+) \to \Gamma(W^-)$. (Thus D_A depends both on the connection A and the Riemannian metric on X.) Given a pair $(A, \psi) \in \mathcal{A}_X(L) \times \Gamma(W^+)$, i.e. A a connection in L and ψ a section of W^+, the monopole equations of Seiberg and Witten [W] are

$$\begin{aligned} D_A\psi &= 0 \\ \rho(F_A^+) &= (\psi \otimes \psi^*)_o \end{aligned} \tag{1}$$

where $(\psi \otimes \psi^*)_o$ is the trace-free part of the endomorphism $\psi \otimes \psi^*$.

The gauge group $\mathrm{Aut}(L) = \mathrm{Map}(X, S^1)$ acts on the space of solutions, and its orbit space is the moduli space $M_X(L)$ whose formal dimension is

$$\dim M_X(L) = \frac{1}{4}(c_1(L)^2 - (3\,\mathrm{sign}(X) + 2\,e(X)). \tag{2}$$

If this formal dimension is nonnegative and if $b^+ > 0$, then for a generic metric on X the moduli space $M_X(L)$ contains no reducible solutions (solutions of the form $(A, 0)$ where A is an anti-self-dual connection on L), and $M_X(L)$ is a compact manifold of the given dimension ([W],[KM]).

The *Seiberg-Witten invariant* for X is the function $SW_X : Spin^c(X) \to \mathbf{Z}$ defined as follows. Let L be a characteristic line bundle. If $\dim M_X(L) < 0$ or is odd, then $SW_X(L)$ is defined to be 0. If $\dim M_X(L) = 0$, the moduli space $M_X(L)$ consists of a finite collection of points and $SW_X(L)$ is defined to be the number of these points counted with signs. These signs are determined by an orientation on $M_X(L)$, which in turn is determined by an orientation on the determinant line $\det(H^0(X;\mathbf{R})) \otimes \det(H^1(X;\mathbf{R})) \otimes \det(H^2_+(X;\mathbf{R}))$. If $\dim M_X(L) > 0$ then we consider the basepoint map

$$\tilde{M}_X(L) = \{\text{solutions}\,(A,\psi)\}/\mathrm{Aut}^0(L) \to M_X(L)$$

where $\mathrm{Aut}^0(L)$ consists of gauge transformations which are the identity on the fiber of L over a fixed basepoint in X. If there are no reducible solutions, the basepoint map is an S^1 fibration, and we denote its euler class by $\beta \in H^2(M_X(L);\mathbf{Z})$. The moduli space $M_X(L)$ represents an integral cycle in the configuration space $B_X(L) = (\mathcal{A}_X(L) \times \Gamma(W^+))/\mathrm{Aut}(L)$, and if $\dim M_X(L) = 2n$, the Seiberg-Witten invariant is defined to be the integer

$$SW_X(L) = \langle \beta^n, [M_X(L)] \rangle.$$

Note that the space $\mathcal{A}_X(L) \times \Gamma(W^+)$ is contractible and $\mathrm{Aut}(L) \cong \mathrm{Map}(X, S^1)$ acts freely on $\mathcal{A}_X(L) \times (\Gamma(W^+) \setminus \{0\})$. Since S^1 is a $K(\mathbf{Z}, 1)$, if we further assume that $H^1(X;\mathbf{R}) = 0$, then the quotient

$$B_X^*(L) = \left(\mathcal{A}_X(L) \times (\Gamma(W^+) \setminus \{0\})\right)/S^1$$

of this action is homotopy equivalent to $\mathbf{CP}^\infty$. So if there are no reducible solutions, we may view $M_X(L) \subset \mathbf{CP}^\infty$. Under these identifications, the class β becomes the standard generator of $H^2(\mathbf{CP}^\infty;\mathbf{Z})$.

If $b^+(X) \geq 2$, the map

$$SW_X : Spin^c(X) \to \mathbf{Z}$$

is a diffeomorphism invariant ([W],[KM]); i.e. $SW_X(L)$ does not depend on the (generic) choice of Riemannian metric on X. To see this, let $\mathcal{C}$ denote the (connected) space of metrics on X and for $g \in \mathcal{C}$ let $M_{X,g}(L)$ denote the corresponding moduli space. As in Donaldson theory (cf. [DK]) consider the parametrized moduli space

$$\mathbf{M}_X(L) = \{(A,\psi,g)|(A,\psi) \in M_{X,g}(L)\} \subset B_X(L) \times \mathcal{C}.$$

We have a Fredholm projection

$$\pi : \mathbf{M}^*_X(L) = \mathbf{M}_X(L) \setminus \{ \text{ reducible solutions } \} \to \mathcal{C}$$

and, in fact, the 'generic' metrics are precisely the regular values of π (an open dense set). Any path γ joining two generic metrics g_0 and g_1 in $\mathcal{C}$ can be perturbed to be transverse to this projection. Then $\pi^{-1}(\gamma)$ is an oriented manifold of dimension $\dim M_X(L)+1$ in $B_X(L)\times[0,1]$, and, provided that none of the moduli spaces $M_{X,\gamma(t)}(L)$ contain reducible solutions, $\pi^{-1}(\gamma)$ is an oriented cobordism between $M_{X,g_0}(L)$ and $M_{X,g_1}(L)$. So

$$SW_X(L,g_0) = \langle\beta^n,[M_{X,g_0}(L)]\rangle = \langle\beta^n,[M_{X,g_1}(L)]\rangle = SW_X(L,g_1)$$

Thus the problem lies with reducible solutions.

The curvature F_A of a connection A on the complex line bundle L is a closed 2-form representing the cohomology class $[F_A] = 2\pi c_1(L) \in H^2(X;\mathbf{R})$. If A is anti-self-dual with respect to a metric g on X then $d^*F_A = *d*F_A = -*dF_A = 0$. Thus if A is g-anti-self-dual, F_A is g-harmonic. Identify $H^2(X;\mathbf{R})$ with the g-harmonic 2-forms, and let

$$H^2(X;\mathbf{R}) = \mathcal{H}^+_g \oplus \mathcal{H}^-_g$$

be its decomposition into the ±1 eigenspaces of the $*$-operator of g. Then A is g-anti-self-dual if and only if $F_A = 2\pi c_1(L) \in \mathcal{H}^-_g$. Since the codimension of the vector subspace $\mathcal{H}^-_g$ of $H^2(X;\mathbf{R})$ is b^+, if $b^+ \geq 1$ the lattice point $c_1(L)$ will not lie on $\mathcal{H}^-_g$ for a generic metric g and L will admit no g-anti-self-dual connections. If $b^+ \geq 2$ the same will be true for paths of metrics as in our argument above. (Rigorous proofs of these facts can be given by using Sard-Smale theory (cf. [DK]).) Thus, if $b^+ \geq 2$, generic paths of generic metrics will admit no reducible solutions of the Seiberg-Witten equations and SW_X will be a diffeomorphism invariant.

For the proof of Theorems 1.1 and 1.2 we are interested in manifolds with $b^+ = 1$ and we need to keep track of the metric in our notation: $SW_X(L,g)$. As above, the line bundle L will admit a g-anti-self-dual connection provided $c_1(L) \in \mathcal{H}^-_g$. Since $b^+ = 1$, $\mathcal{H}^+_g \equiv \mathbf{R}$ so that up to scale there is a unique g-self-dual harmonic 2-form ω_g. Since $\mathcal{H}^+_g$ and $\mathcal{H}^-_g$

are L^2-orthogonal, L admits a g-anti-self-dual connection if and only if $c_1(L) \cdot \omega_g = 0$. The self-dual harmonic 2-form

$$\omega_g \in \mathbf{P}(\{\alpha \in H^2(X;\mathbf{R}) | \alpha \cdot \alpha > 0\})$$

is called the *period point* of the metric g. Reducible solutions to the Seiberg-Witten equations appear only for those metrics g whose period points ω_g lie in the hyperplane $c_1(L)^{\perp}$. This hyperplane separates $\mathbf{P}(\{\alpha \in H^2(X;\mathbf{R}) | \alpha \cdot \alpha > 0\})$ into two chambers given by the inequalities $c_1(L) \cdot \omega_g > 0$ and $c_1(L) \cdot \omega_g < 0$. Any two generic metrics g_0 and g_1 whose period points lie in the same chamber can be connected by a path of metrics in that chamber, and the argument above shows $SW_X(L, g_0) = SW_X(L, g_1)$. Thus for a fixed characteristic line bundle L, as a function of g the invariant $SW_X(L, g)$ takes on at most two possible values.

To understand what happens as the period points of a path γ of metrics cross the hyperplane $c_1(L)^{\perp}$ transversely at a single point we utilize the Kuranishi model of the parametrized moduli space $\pi^{-1}(\gamma)$ (cf. [DK], [D1]). In case the formal dimension $\dim M_X(L) = 0$, it models the one-parameter family of moduli spaces $M_{X,g_t}(L)$ near a reducible solution $(A, 0) \in M_{X,g_0}(L)$ as the zero set of the map $z \to |z|^2 + t$ from $\mathbf{C} \to \mathbf{R}$. Thus

$$SW_X(L, g_{-1}) = SW_X(L, g_{+1}) \pm 1$$

depending on the direction that the path crosses the hyperplane. The same argument extends to the case where $\dim M_X(L) > 0$ resulting in a simple 'wall-crossing formula'.

3. The Proofs of Theorems 1.1 and 1.2

We begin by proving Theorem 1.1. Suppose that there is a smooth regular immersion $S^2 \to \mathbf{CP}^2$ which has p positive and n negative double points. Further assume that this immersion represents the class dH and that

$$p = \frac{(d-1)(d-2)}{2} - 1 = \frac{d^2 - 3d}{2} \tag{3}$$

and look for a contradiction. (In case there are fewer positive double points just increase the number of positive double points by connect sums with 2-spheres representing $0 \in H_2$ immersed with a pair of 'cancelling' double points.)

The first step is to convert the immersion in $\mathbf{CP}^2$ into an embedding in $\mathbf{CP}^2 \#(p + n)\overline{\mathbf{CP}}^2$ by blowing up. Let E denote the generator of $H_2(\overline{\mathbf{CP}}^2; \mathbf{Z})$ represented by the exceptional curve. Blowing up at a double point of our immersion will remove the double point in the connected sum with $\overline{\mathbf{CP}}^2$. This process adds one copy of $\pm E$ to each sheet of the immersion at this point. If the double point is positive, both copies are $-E$, and if the double point is negative, then one copy is E and the other is $-E$. Thus if $\Sigma \subset X$ is represented by an immersed 2-sphere with r double points, then by blowing up at a double point we obtain an immersed sphere with $r - 1$ double points in $X \# \overline{\mathbf{CP}}^2$ representing

$\Sigma - 2E \in H_2(X\#\overline{\mathbf{CP}}^2; \mathbf{Z})$ if the double point is positive, and representing Σ if the double point is negative.

Let X be the rational surface obtained by blowing up $\mathbf{CP}^2$ a number $N = p + n + q$ times so that

(a) The homology class

$$\Sigma = dH - 2\sum_{i=1}^{p} E_i - \sum_{j=1}^{q} E_j$$

is represented by an embedded 2-sphere in the rational surface $X = \mathbf{CP}^2\#N\overline{\mathbf{CP}}^2$, and

(b) q is chosen so that the self intersection $\Sigma \cdot \Sigma = d^2 - 4p - q < 0$.

The assumption (3) implies $\Sigma \cdot \Sigma = 6d - d^2 - q$. Now let

$$K = 3H - \sum_{i=1}^{N} E_i$$

which is the negative of the canonical class of the rational surface X. Since $K = c_1(X) = (3\,\mathrm{sign} + 2\,\mathrm{e})(X)$, it follows from the dimension formula (2) that $\dim M_K = 0$. For the characteristic line bundle $K - 2\Sigma$, the dimension formula (2) shows

$$\dim M_{K-2\Sigma} = \dim M_K + \Sigma \cdot \Sigma - \Sigma \cdot K.$$

However by (3),

$$\Sigma \cdot K = 3d - 2p - q = 6d - d^2 - q = \Sigma \cdot \Sigma$$

so also $\dim M_{K-2\Sigma} = 0$. Note that if we were to change assumption (3) by increasing the right hand side (so that there should not be a contradiction), then the resultant formal dimension of $M_{K-2\Sigma}$ would be negative and the following discussion would not apply.

Since $b^+(X) = 1$, each of the bundles determines a hyperplane $K^\perp$ and $(K - 2\Sigma)^\perp$ and corresponding chambers of the projectivization of the positive cone of $H^2(X;\mathbf{R})$. We next wish to determine the Seiberg-Witten invariants $SW_X(K, g)$ and $SW_X(K - 2\Sigma, g)$. Since the Fubini-Study metric on $\mathbf{CP}^2$ (and on $\overline{\mathbf{CP}}^2$) has positive scalar curvature, we can glue these together on the connected sum, as explained in [GL], to obtain a metric of positive scalar curvature on X. If we take a sequence of metrics $\{g_t\}$ shrinking the size of the necks in the connected sum to zero, we obtain the wedge of the Fubini-Study metrics on $\mathbf{CP}^2 \amalg \overline{\mathbf{CP}}^2 \amalg \cdots \amalg \overline{\mathbf{CP}}^2$. The limit of the period points ω_{g_t} is a harmonic self-dual 2-form on $\mathbf{CP}^2 \amalg \overline{\mathbf{CP}}^2 \amalg \cdots \amalg \overline{\mathbf{CP}}^2$; thus up to scale, it is H. Write g_+ for g_t, t large; so we see that ω_{g_+} is approximately equal to H. Since g_+ has positive scalar curvature, $SW_X(K, g_+) = 0$ and $SW_X(K - 2\Sigma, g_+) = 0$. But $K \cdot \omega_{g_+} \sim K \cdot H = 3 > 0$, and $(K-2\Sigma)\cdot\omega_{g_+} \sim (K-2\Sigma)\cdot H = 3-2d < 0$, provided $d \geq 2$. Since Theorems 1.1 and 1.2 are trivial for $d=1$, we assume that $d \geq 2$. The wall-crossing formula then implies:

$$SW_X(K,g) = \begin{cases} 0, & \text{if } K\cdot\omega_g > 0 \\ \pm 1, & \text{if } K\cdot\omega_g < 0 \end{cases} \tag{4}$$

$$SW_X(K-2\Sigma,g) = \begin{cases} \pm 1, & \text{if } K\cdot\omega_g > 0 \\ 0, & \text{if } K\cdot\omega_g < 0. \end{cases} \tag{5}$$

To complete our argument we consider a tubular neighborhood $U \subset X$ of the smoothly embedded 2-sphere representing the homology class Σ. The boundary of U is the lens space $L(p,-1)$. We write $X = X_0 \cup L(p,-1)\times(-\epsilon,\epsilon)\cup U$. The neighborhood U admits metrics of positive scalar curvature; so we can obtain a family of generic metrics on X: $g_r = g_0 \cup g_{L,r} \cup g_U$ where $g_{L,r}$ and g_U have positive scalar curvature, and $g_{L,r}$ makes the neck isometric to $L(p,-1)\times(-r,r)$. Let $X_0^+ = X_0 \cup L(p,1)\times[0,\infty)$ and $U^+ = U \cup L(p,-1)\times[0,\infty)$. The Mayer-Vietoris sequence gives an isomorphism $j^* : H^2(X;\mathbf{R}) \cong H^2(X_0^+;\mathbf{R})\oplus H^2(U^+;\mathbf{R})$. Let $j^*(K) = K_0 + K_U$. An argument similar to the one given above shows that for the $b^+=1$ (metrically) cylindrical end manifold X_0^+, a metric admits an anti-self-dual connection on the line bundle with $c_1 = K_0$ if and only if its period point is orthogonal to K_0, and for a generic metric this condition does not hold. Thus we may choose g_0 so that

$$C = \lim_{r\to\infty} \omega_{g_r|_{X_0^+}}\cdot K_0 \neq 0.$$

Since the necks have positive scalar curvature, there is a gluing theory for obtaining solutions to the Seiberg-Witten equations on X from solutions on X_0^+ and U^+, and this theory parallels the gluing theory for solutions of the anti-self-duality equations on a connected sum. (See [W] and [J].)

Since the neighborhood U^+ has positive scalar curvature, the only solution of the Seiberg-Witten equations for K is the reducible solution $(A,0)$ where A is an anti-self-dual connection on $K|_{U^+}$. Similarly, the only solution on U^+ for $K-2\Sigma$ is the reducible solution $(A',0)$. Note that the line bundles K and $K-2\Sigma$ agree on X_0^+ and so we may identify the moduli spaces $M_{X_0^+}(K) = M_{X_0^+}(K-2\Sigma) = M_0$. Since

$$0 = \dim M_X(K) = \dim M_{X_0^+}(K) + \dim M_{U^+}(K) + 1$$

and

$$\dim M_X(K-2\Sigma) = 0,$$

the formal dimensions of the moduli spaces $M_{U^+}(K)$ and $M_{U^+}(K-2\Sigma)$ are equal. In fact an index calculation using the Atiyah-Patodi-Singer formula shows that both dimensions are equal to -1. It follows from the gluing theory that for the metric g_r, with r large

$$M_{X,g_r}(K) \cong M_0 \times \{(A,0)\} \cong M_0 \times \{(A',0)\} \cong M_{X,g_r}(K-2\Sigma).$$

Hence, counting points in these 0-dimensional moduli spaces we obtain for large r,

$$SW_X(K, g_r) \equiv SW_X(K - 2\Sigma, g_r) \pmod 2. \tag{6}$$

However, since $\Sigma\cdot\Sigma < 0$, the intersection form of U is negative definite, so $\lim_{r\to\infty} \omega_{g_r}|_{U^+} = 0$. This means that

$$\lim_{r\to\infty} \omega_{g_r} \cdot (K - 2\Sigma) = \lim_{r\to\infty} \omega_{g_r} \cdot K = C.$$

Hence for large r, $\omega_{g_r} \cdot (K - 2\Sigma)$ and $\omega_{g_r} \cdot K$ both have the same sign as C. This contradicts (6) and (4) and (5) and completes the proof of Theorem 1.1.

The proof of Theorem 1.2 follows easily by examining the above proof.

4. Outline of the Proof of Theorem 1.4

If L is a characteristic line bundle on X, then, for a nonnegative integer k, $L \pm (2k+1)E$ is a characteristic line bundle on $X\#\overline{\mathbf{CP}}^2$. Computing dimensions, we have

$$\dim M_{X\#\overline{\mathbf{CP}}^2}(L \pm (2k+1)E) = \dim M_X(L) - k(k+1).$$

If $\dim M_X(L) = 0$, then

$$M_{X\#\overline{\mathbf{CP}}^2}(L \pm E) = [\tilde{M}_X(L) \times \tilde{M}_{\overline{\mathbf{CP}}^2}(\pm E)]/S^1.$$

Since $\overline{\mathbf{CP}}^2$ has a metric of positive scalar curvature, the only solutions on E are the reducible solutions $(A_{\pm E}, 0)$. Thus $M_{X\#\overline{\mathbf{CP}}^2}(L \pm E) \cong [\tilde{M}_X(L) \times \{(A_{\pm E}, 0)\}]/S^1 \cong M_X(L) \times \{(A_{\pm E}, 0)\} \cong M_X(L)$ with (A, ψ) being identified with $(A\#A_{\pm E}, \psi + 0)$. Thus, when $\dim M_X(L) = 0$, $SW_X(L) = SW_{X\#\overline{\mathbf{CP}}^2}(L \pm E)$.

If $\dim M_X(L) > 0$, we need to compute using the basepoint fibration. So assume $\dim M_X(L) = 2d$, then $SW_X(L) = \langle \beta^d, [M_X(L)] \rangle$. Let k be an integer so that $k(k+1) < d$; hence $\dim M_{X\#\overline{\mathbf{CP}}^2}(L \pm (2k+1)E) = 2d - k(k+1) \geq 0$. To simplify notation, let $L_k = L \pm (2k+1)E$. If $k > 0$ it is no longer the case that $\tilde{M}_{X\#\overline{\mathbf{CP}}^2}(L_k) \cong \tilde{M}_X(L) \times \tilde{M}_{\overline{\mathbf{CP}}^2}(\pm(2k+1)E)$; although it still is the case that $\tilde{M}_{\overline{\mathbf{CP}}^2}(\pm(2k+1)E)$ is a point. For this reducible solution over $\overline{\mathbf{CP}}^2$ there are complications arising from H^2 of the deformation complex which gives rise to the obstruction bundle for gluing in reducible solutions. Formally, this is the same as for Donaldson theory (cf. see Theorem 4.53 of [D2]). To study solutions of the form $(A, \psi)\#(A_{\pm(2k+1)E}, 0)$, the model is as follows. There is a $\mathbf{C}^{\frac{k(k+1)}{2}}$-fibration $\tilde{\xi}$ over $\tilde{M}_X(L) \times \tilde{M}_{\overline{\mathbf{CP}}^2}(\pm(2k+1)E) = \tilde{M}_X(L) \times \{(A_{\pm(2k+1)E}, 0)\}$ with an S^1-equivariant section Φ so that

$$\Phi^{-1}(0) = \tilde{M}_{X\#\overline{\mathbf{CP}}^2}(L_k).$$

The bundle $\tilde{\xi}$ is pulled back from the (trivial) bundle over the point $(A_{\pm(2k+1)E}, 0)$ whose total space is the cokernel of the twisted Dirac operator on $\overline{\mathbf{CP}}^2$. Thus $\tilde{\xi}$ is trivial (but not equivariantly). Thus its quotient

$$\xi = \tilde{M}_X(L) \times_{S^1} \mathbf{C}^{\frac{k(k+1)}{2}}$$

is a $\mathbf{C}^{\frac{k(k+1)}{2}}$-bundle over $M_X(L)$ with a section Φ so that $\Phi^{-1}(0) = M_{X\#\overline{\mathbf{CP}}^2}(L_k)$.

The bundle ξ is clearly associated with the basepoint fibration; hence $\xi = \beta^{\frac{k(k+1)}{2}}$ (where β is the line bundle associated to the basepoint fibration over $M_X(L)$). Thus, viewing $M_X(L)$ as a subset of $B_{X\#\overline{\mathbf{CP}}^2}(L_k)$ (by identifying (A,ψ) with $(A\#A_{\pm(2k+1)E}, \psi\#0)$) we see that the homology class $[\Phi^{-1}(0)]$ is just the zero set of the basepoint induced fibration ξ over $M_X(L)$. Recalling $\dim M_{X\#\overline{\mathbf{CP}}^2}(L_k)/2 = d - \frac{k(k+1)}{2}$, we have:

$$\langle \beta^{\frac{\dim M(L_k)}{2}}, [M_{X\#\overline{\mathbf{CP}}^2}(L_k)]\rangle = \langle \beta^{d-\frac{k(k+1)}{2}}, [\Phi^{-1}(0)] >=< \beta^d, [M_X(L)]\rangle$$

so that

$$SW_{X\#\overline{\mathbf{CP}}^2}(L \pm (2k+1)E) = SW_X(L).$$

□

5. The Proof of Theorem 1.3

Lemma 5.1. *Suppose that X is a smooth 4-manifold with $b^+ > 1$, and S is an essential embedded sphere in X of nonnegative self-intersection. Then $SW_X : Spin^c(X) \to \mathbf{Z}$ is the zero map.*

Proof. If $S \cdot S > 0$, then S has a tubular neighborhood with $b^+ = 1$ and boundary a lens space. Furthermore, the complement of S also has $b^+ > 0$; so the proof follows exactly as for the connected sum theorem. (Cf. [W].)

If $S \cdot S = 0$, then form the connected sum $\overline{X} = X\#\overline{\mathbf{CP}}^2$ with exceptional curve E. For each positive integer n the class $nS + E$ has self-intersection -1 and is represented by an embedded sphere in $\overline{X}$. Let U_n be a regular neighborhood of this sphere, and let $Y_n = (\overline{X} \setminus U_n) \cup B^4$. Thus $\overline{X} = Y_n\#\overline{\mathbf{CP}}^2$ where $E_n = nS + E$ is the exceptional class. Assume that there is a characteristic line bundle L on X with $SW_X(L) \neq 0$. Then by Theorem 1.4, $SW_{\overline{X}}(L+E) \neq 0$. But $L + E = L_n + E_n$ where, again by Theorem 1.4, $L_n = L - nS$ is a characteristic line bundle on Y_n and $SW_{Y_n}(L_n) \neq 0$. Using Theorem 1.4 one last time, we see that $SW_{\overline{X}}(L_n - E_n) \neq 0$, and $L_n - E_n = L - E - 2nS$. If S is homologically nontrivial, this process gives infinitely many characteristic line bundles $\{L - E - 2nS\}$ on $\overline{X}$ with nontrivial Seiberg-Witten invariants, and this is a contradiction [W]. □

The proof of Theorem 1.3 follows from Theorem 1.4 and

Lemma 5.2 (Fundamental Lemma). *Suppose X is a smooth 4-manifold with an embedded sphere S with self-intersection $-r < 0$. Let L be a characteristic line bundle with $SW_X(L) \neq 0$ and write*

$$|S \cdot L| = kr + R$$

with $0 \leq R \leq r - 1$. If $k > 0$, then

$$SW_X(L) = \begin{cases} SW_X(L + 2S) & \text{if } L \cdot S > 0 \\ SW_X(L - 2S) & \text{if } L \cdot S < 0. \end{cases}$$

Proof. Note that the hypothesis $k > 0$ holds if and only if $S \cdot S + |S \cdot L| \geq 0$ (which is a violation of the ordinary adjunction formula).

Suppose $S \cdot L > 0$ so that $S \cdot L = kr + R$. Then just as in the proof of Theorem 1.4, the regular neighborhood N of the sphere S has a metric of positive scalar curvature so that there is just the reducible solution $(A_N, 0)$ for $L|_N$. Write

$$X = X_0 \cup N.$$

Again, $(L+2S)|_{X_0} = L_{X_0}$ and there is just the reducible solution $(A_{N'}, 0)$ for $(L+2S)|_N$. The exact same proof as for the blowup formula Theorem 1.4 shows $SW_X(L') = SW_X(L)$.

The proof when $S \cdot L < 0$ is the same with $L + 2S$ replaced by $L - 2S$. $\square$

To prove Theorem 1.3 suppose that $x \in H_2(X; \mathbf{Z})$ is represented by an immersed sphere with p positive and n negative double points. Let L be a characteristic line bundle over X with $SW_X(L) \neq 0$ and with $\dim M_X(L) = \sum_{i=1}^r \ell_i(\ell_i + 1)$. Suppose first that $r \leq p$. For simplicity, assume that $x \cdot L \geq 0$. (If $x \cdot L \leq 0$, a similar argument will apply.) Then in $\overline{X} = X \# (p+n)\overline{\mathbf{CP}}^2$, $\overline{x} = x - 2\sum_{j=1}^p E_i$ is represented by an embedded sphere. Let $\overline{L} = L + \sum_{j=1}^r (2\ell_j + 1)E_j + E_{r+1} + \cdots + E_{p+n}$. Apply the Fundamental Lemma 5.2: either $\overline{x}^2 + \overline{x} \cdot \overline{L} \leq -2$ or $SW_{\overline{X}}(\overline{L} + 2\overline{x}) = SW_{\overline{X}}(\overline{L}) \neq 0$. Now

$$\overline{x}^2 = x^2 - 4p$$

$$\overline{x} \cdot \overline{L} = x \cdot L + 4\sum_{j=1}^r \ell_j + 2p.$$

So in the first case

$$x^2 + x \cdot L + 4\sum_{j=1}^r \ell_j \leq 2p - 2.$$

Otherwise

$$SW_{\overline{X}}(\overline{L} + 2\overline{x}) \neq 0.$$

Furthermore, the blowup formula, Theorem 1.4, says

$$SW_{\overline{X}}(\overline{L} + 2\overline{x}) = SW_X(L + 2x).$$

In case $p < r$, let $\overline{X}$ be X blown up $m = \max\{p+n, r\}$ times, and let $\overline{x} = x - 2\sum_{j=1}^{p} E_i - \sum_{j=p+1}^{r} E_i$ and $\overline{L} = L + \sum_{j=1}^{r}(2\ell_j + 1)E_j + E_{r+1} + \cdots + E_m$. Then the above proof goes through. □

Call a characteristic line bundle with nontrivial Seiberg-Witten invariant a *Seiberg-Witten class.* Further, X is said to have *Seiberg-Witten simple type* if $\dim M_X(L) = 0$ for any Seiberg-Witten class L. It is an open question whether all 4-manifolds with $b^+ > 1$ have Seiberg-Witten simple type. Theorem 1.3 can be used to give criteria for a manifold to have Seiberg-Witten simple type. For example, if X contains an embedded sphere S with $S \cdot S = -2$ and such that $S \cdot L = 0$ for every Seiberg-Witten class L, then X has Seiberg-Witten simple type. For, since $(L \pm 2S) \cdot S \neq 0$, $L \pm 2S$ can't be a Seiberg-Witten class. Thus the first case of Theorem 1.3 must hold, and this implies $\dim M_X(L) = 0$.

Iterations of Theorem 1.3 also give interesting results. For simplicity, assume $x^2 > 0$ and choose the Seiberg-Witten class L so that $x \cdot L > 0$ (otherwise choose $-L$). Then either

$$2p - 2 \geq x^2 + x \cdot L + 4\sum_{i=1}^{r} \ell_i, \quad p \geq r$$

$$2p - 2 \geq x^2 + x \cdot L + 4\sum_{i=1}^{p} \ell_i + 2\sum_{i=p+1}^{r} \ell_i, \quad p < r$$

or $L + 2x$ is also a Seiberg-Witten class. Note then that $\dim M_X(L+2x) - \dim M_X(L) = x^2 + x \cdot L > 0$. In this latter case apply Theorem 1.3 with L replaced by $L + 2x$ to obtain the stronger adjunction formula

$$2p - 2 \geq 3x^2 + x \cdot L + 4\sum_{i=1}^{r} \ell_i', \quad p \geq r$$

$$2p - 2 \geq 3x^2 + x \cdot L + 4\sum_{i=1}^{p} \ell_i' + 2\sum_{i=p+1}^{r} \ell_i', \quad p < r$$

(where $4\sum_{i=1}^{r} \ell_i' > 4\sum_{i=1}^{r} \ell_i$) or else $L + 4x$ will also be a basic class. This process must terminate since there are only finitely many Seiberg-Witten classes [W]. So in this situation we have an even stronger adjunction formula for x and L. Thus, for example, if x is represented by an immersed sphere with $2p - 2 = x^2$, then X has Seiberg-Witten simple type. There are other variations on this theme that can be useful in showing that a manifold X has Seiberg-Witten simple type.

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