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# The blowup formula for Donaldson invariants

By RONALD FINTUSHEL and RONALD J. STERN\*

## 1. Introduction

Since their introduction in 1984 [4], the Donaldson invariants of smooth 4-manifolds have remained as mysterious as they have been useful. However, in the past year there has been a surge of activity pointed at comprehension of the structure of these invariants [10], [6]. One key to these advances and to future insights lies in understanding the relation of the Donaldson invariants of a 4-manifold  $X$  and those of its blowup  $\hat{X} = X \# \overline{\mathbf{CP}}^2$ . It is the purpose of this paper to present such a blowup formula. This formula is independent of  $X$  and is given in terms of an infinite series

$$B(x, t) = \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!}$$

which is calculated in Section 4 below. This formula has been the target of much recent work. The abstract fact that there exists such a formula which is independent of  $X$  was first proved by C. Taubes using techniques of [16]. J. Bryan [3] and P. Ozsvath [14] have independently calculated the coefficients through  $B_{10}(x)$ . Quite recently, J. Morgan and Ozsvath have announced a scheme which can recursively compute all of the  $B_k(x)$ . The special case of the blowup formula for manifolds of “simple type” (see §5 below) was first given by P. Kronheimer and T. Mrowka. However, none of the techniques in these cases approach the simplicity of that offered here.

Before presenting the formula, we shall first establish notation for the Donaldson invariants of a simply connected 4-manifold  $X$  with  $b^+ > 1$  and odd. (The hypothesis of simple connectivity is not necessary, but makes the exposition easier.) An orientation of  $X$ , together with an orientation of  $H_+^2(X; \mathbf{R})$  is

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called a *homology orientation* of  $X$ . Such a homology orientation determines the (SU(2)) Donaldson invariant, a linear function

$$D = D_X: \mathbf{A}(X) = \text{Sym}_*(H_0(X) \oplus H_2(X)) \rightarrow \mathbf{R}$$

which is a homology orientation-preserving diffeomorphism invariant. Here

$$\mathbf{A}(X) = \text{Sym}_*(H_0(X) \oplus H_2(X))$$

is viewed as a graded algebra where  $H_i(X)$  has degree  $\frac{1}{2}(4-i)$ . We let  $x \in H_0(X)$  be the generator [1] corresponding to the orientation. Then, as usual, if  $a + 2b = d > \frac{3}{4}(1 + b_X^+)$  and  $\alpha \in H_2(X)$ ,

$$D(\alpha^a x^b) = \langle \mu(\alpha)^a \nu^b, [\mathcal{M}_X^{2d}] \rangle$$

where  $[\mathcal{M}_X^{2d}]$  is the fundamental class of the (compactified)  $2d$ -dimensional moduli space of anti-self-dual connections on an SU(2) bundle over  $X$ ,  $\mu: H_i(X) \rightarrow H^{4-i}(\mathcal{B}_X^*)$  is the canonical map to the cohomology of the space of irreducible connections on that bundle [4], and  $\nu = \mu(x)$ . The extension of the definition to smaller  $d$  is given in [13] (and is accomplished, in fact, from the knowledge of the lowest coefficient in the SO(3) blowup formula). Since an SU(2) bundle  $P$  over  $X$  has a moduli space of dimension

$$\dim \mathcal{M}_X(P) = 8c_2(P) - 3(1 + b_X^+),$$

it follows that such moduli spaces  $\mathcal{M}_X^{2d}$  can exist only for  $d \equiv \frac{1}{2}(1 + b_X^+) \pmod{4}$ . Thus the Donaldson invariant  $D$  is defined only on elements of  $\mathbf{A}(X)$  whose total degree is congruent to  $\frac{1}{2}(1 + b_X^+) \pmod{4}$ . By definition,  $D$  is 0 on all elements of other degrees.

We can now state the blowup formula. Let  $\hat{X} = X \# \overline{\mathbf{CP}}^2$  and let  $e \in H_2(X)$  denote the homology class of the exceptional divisor. Since  $b_X^+ = b_{\hat{X}}^+$ , the corresponding Donaldson invariants  $D = D_X$  and  $\hat{D} = D_{\hat{X}}$  have their (possible) nonzero values in the same degrees mod 4. We first show that there are polynomials  $B_k(x)$  satisfying

$$\hat{D}(e^k z) = D(B_k(x) z)$$

for all  $z \in \mathbf{A}(X)$  and then define the formal power series  $B(x, t)$  as above. Our result is that

$$B(x, t) = e^{-\frac{t^2 x}{6}} \sigma_3(x, t),$$

where, as a function of  $t$ ,  $\sigma_3$  is a particular quasi-periodic Weierstrass sigma-function [1] associated to the  $\wp$ -function which satisfies the differential equation

$$(y')^2 = 4y^3 - g_2 y - g_3$$

where

$$g_2 = 4 \left( \frac{x^2}{3} - 1 \right), \quad g_3 = \frac{8x^3 - 36x}{27}.$$

There are also Donaldson invariants associated to  $\mathrm{SO}(3)$  bundles  $V$  over  $X$ . To define these invariants one needs, along with a homology orientation of  $X$ , an integral lift of  $w_2(V)$ . If  $c \in H_2(X; \mathbf{Z})$  is the Poincaré dual of the lift, the invariant is denoted  $D_c$  or  $D_{X,c}$  if the manifold  $X$  is in doubt. The invariant  $D_c$  is nonzero only in degrees congruent to  $-c \cdot c + \frac{1}{2}(1 + b^+)$  (mod 4). If  $c' \equiv c \pmod{2}$ , then

$$D_{c'} = (-1)^{\left(\frac{c'-c}{2}\right)^2} D_c.$$

The  $\mathrm{SO}(3)$  blowup formula states that there are polynomials  $S_k(x)$  such that

$$\hat{D}_e(e^k) = D(S_k(x)),$$

and if

$$S(x, t) = \sum_{k=0}^{\infty} S_k(x) \frac{t^k}{k!},$$

then

$$S(x, t) = e^{-\frac{t^2 x}{6}} \sigma(x, t)$$

where, as a function of  $t$ ,  $\sigma(x, t)$  is the standard Weierstrass sigma-function [1] associated to  $\wp$ . The coefficients  $S_k(x)$  for  $k \leq 7$  were earlier computed by T. Leness [12].

The discriminant of the cubic equation  $4y^3 - g_2y - g_3 = 0$  turns out to be  $\frac{x^2-4}{4}$ . Thus, when (viewed as a function on  $\mathbf{A}(X)$ )  $D(x^2 - 4) = 0$ , the Weierstrass sigma-functions degenerate to elementary functions, and the blowup formula can be restated in terms of these functions. This is done in the final section. It is interesting to note that the condition  $D(x^2 - 4) = 0$  is the *simple type* condition introduced by Kronheimer and Mrowka [10].

Our formulas are proved by means of a simple relation satisfied by  $D(\tau^4 z)$  where  $\tau \in H_2(X; \mathbf{Z})$  is represented by an embedded 2-sphere of self-intersection  $-2$ . When this relation is applied to  $\tau = e_1 - e_2$ , the difference of the two exceptional classes of the double blowup  $X \# 2\overline{\mathbf{CP}}^2$ , one obtains a differential equation for  $B(x, t)$ . Solving this equation gives our formulas.

## 2. Some relations among Donaldson invariants

The key to the blowup formula lies in a few simple relations which are useful for evaluating Donaldson invariants on classes represented by embedded spheres of self-intersection  $-2$  and  $-3$ . We begin by studying the behavior of the Donaldson invariant of a 4-manifold with a homology class  $\tau$  represented by an embedded 2-sphere  $S$  of self-intersection  $\tau \cdot \tau = -2$ . Let  $\langle \tau \rangle^\perp$  denote  $\{\alpha \in H_2(X) \mid \tau \cdot \alpha = 0\}$  and let

$$\mathbf{A}_X(\tau^\perp) = \mathbf{A}(\tau^\perp) = \mathrm{Sym}_*(H_0(X) \oplus \langle \tau \rangle^\perp).$$

**THEOREM 2.1** (Ruberman [15]). *Suppose that  $\tau \in H_2(X; \mathbf{Z})$  with  $\tau \cdot \tau = -2$  is represented by an embedded sphere  $S$ . Then for  $z \in \mathbf{A}(\tau^\perp)$ , we have  $D(\tau^2 z) = 2 D_\tau(z)$ .*

*Proof.* Write  $X = X_0 \cup N$  where  $N$  is a tubular neighborhood of  $S$ , and note that  $\partial N$  is the lens space  $L(2, -1)$ . Each  $\mathrm{SU}(2)$  representation of  $\pi_1(\partial N) = \mathbf{Z}_2$  is conjugate to a representation into  $U(1)$ , and so the character variety of  $\mathrm{SU}(2)$  representations mod conjugacy may be identified with  $H^2(\partial N; \mathbf{Z})/\{\pm 1\} = \{\zeta^m \mid m \in \mathbf{Z}_2\}$  where  $\zeta$  is the nontrivial character. Since  $b_{X_0}^+ > 0$ , generically there are no reducible anti-self-dual connections on  $X_0$ . However, since  $b_N^+ = 0$ , there are nontrivial reducible anti-self-dual connections arising from complex line bundles  $\lambda^m$ ,  $m \in \mathbf{Z}$ , where  $\langle c_1(\lambda), \tau \rangle = -1$ . The corresponding moduli spaces  $\mathcal{M}_N(\lambda^m \oplus \bar{\lambda}^m)$  have dimensions  $4m^2 - 3$  and have boundary values  $\zeta^m$ .

Since  $\langle \tau \rangle^\perp = H_2(X_0)$ , we need to evaluate the Donaldson invariant on two copies of  $\tau$  and classes in  $H_2(X_0)$ . After cutting down moduli spaces by intersecting with transverse divisors representing the images under  $\mu$  of these classes in  $H_2(X_0)$  and using the given homology orientation, we may assume, without loss of generality, that there are no such classes and that we are working with a 4-dimensional moduli space  $\mathcal{M}_X$ . Let  $V_1$  and  $V_2$  be divisors representing  $\mu(\tau)$ , coming from general positioned surfaces in  $N$ . The Donaldson invariant is the signed intersection number

$$D(\tau^2) = \#(\mathcal{M}_X \cap V_1 \cap V_2).$$

A standard dimension counting argument (cf. [4]) shows that if we choose a metric on  $X$  with long enough neck length,  $\partial N \times [0, T]$ , then all the intersections take place in a neighborhood  $\mathcal{U}$  of the grafted moduli space  $\mathcal{M}_{X_0}[\zeta] \# \{A_\lambda\}$ , where  $A_\lambda$  is the reducible anti-self-dual connection on  $\lambda \oplus \bar{\lambda}$ , and  $\mathcal{M}_{X_0}[\zeta]$  is the 0-dimensional cylindrical end moduli space on  $X_0$  consisting of anti-self-dual connections which decay exponentially to the boundary value  $\zeta$ . Let  $m_{X_0}$  be the signed count of points in  $\mathcal{M}_{X_0}[\zeta]$ . A neighborhood of  $A_\lambda$  in the moduli space  $\mathcal{M}_N(\lambda \oplus \bar{\lambda})$  is diffeomorphic to  $(\mathbf{C} \times_{S^1} \mathrm{SO}(3))/\mathrm{SO}(3) \cong \mathbf{C}/S^1 \cong [0, \infty)$ . Here  $S^1$  acts on  $\mathrm{SO}(3)$  so that  $\mathrm{SO}(3)/S^1 = S^2$  and on  $\mathbf{C}$  with weight  $-2$ . Thus the neighborhood  $\mathcal{U}$  is

$$(\tilde{\mathcal{M}}_{X_0}[\zeta] \times (\mathbf{C} \times_{S^1} \mathrm{SO}(3)))/\mathrm{SO}(3)$$

where “ $\tilde{\mathcal{M}}_{X_0}[\zeta]$ ” denotes the based moduli space.

Now  $\tilde{V}_1 \cap (\mathbf{C} \times_{S^1} \mathrm{SO}(3)) = \{0\} \times_{S^1} \mathrm{SO}(3)$ , and the intersection of  $V_1$  with all of  $\mathcal{M}_X$  is

$$(\mathcal{M}_{X_0}[\zeta] \times (\{0\} \times_{S^1} \mathrm{SO}(3)))/\mathrm{SO}(3) = \Delta.$$

Fix a point  $p \in \mathcal{M}_{X_0}[\zeta]$ , let  $\mathrm{SO}(3) \cdot p$  denote its orbit in  $\tilde{\mathcal{M}}_{X_0}[\zeta]$ , and let

$$\Delta_p = \mathrm{SO}(3) \cdot p \times (\{0\} \times_{S^1} \mathrm{SO}(3))/\mathrm{SO}(3) \cong S^2.$$

Identify  $\Delta_p$  with a transversal in  $\tilde{\Delta}_p$  and compute the intersection number  $\tilde{V}_2 \cdot \Delta_p = \iota_p$ . Since  $\iota_p$  is independent of  $p \in \mathcal{M}_{X_0}[\zeta]$ , we have  $D(\tau^2) = \iota_p \cdot m_{X_0}$ . The constant  $\iota_p$  is computed in [7] as follows. Note that  $\Delta_p = \{0\} \times_{S^1} \mathrm{SO}(3) \subset \mathbf{C} \times_{S^1} \mathrm{SO}(3)$  is a zero-section of the  $c_1 = -2$  complex line bundle over  $S^2$  and  $\tilde{V}_2$  is another section. Thus  $\tilde{V}_2 \cdot \Delta_p = -2$ , and so  $D(\tau^2) = -2 m_{X_0}$ .

To identify the relative invariant  $m_{X_0}$ , view  $\mathcal{M}_{X_0}[\zeta]$  as  $\mathcal{M}_{X_0,0}[\mathrm{ad}(\zeta)]$ , an  $\mathrm{SO}(3)$  moduli space. Since  $\mathrm{ad}(\zeta)$  is the trivial  $\mathrm{SO}(3)$ -representation, we may graft connections in  $\mathcal{M}_{X_0,0}[\mathrm{ad}(\zeta)]$  to the trivial  $\mathrm{SO}(3)$  connection over  $N$ , and since  $b_N^+ = 0$ , there is no obstruction to doing this. We obtain an  $\mathrm{SO}(3)$  moduli space over  $X$  corresponding to an  $\mathrm{SO}(3)$  bundle over  $X$  with  $w_2$  Poincaré dual to  $\tau$ . (This is the unique nonzero class in  $H^2(X; \mathbf{Z}_2)$  which restricts trivially to both  $N$  and  $X_0$ .) Thus for  $z \in \mathbf{A}(\tau^\perp)$ , we have  $D(\tau^2 z) = \pm 2 D_\tau(z)$ . (Note that since  $\tau \cdot \tau = -2$ , we have  $D_{-\tau} = D_\tau$ .)

To determine the sign in this equation, we need to compare orientations on the moduli spaces which are involved. Let  $A_0 \in \mathcal{M}_{X_0}[\zeta]$ . The way that a sign is attached to this point is described in [5], [9]. By addition and subtraction of instantons,  $A_0$  is related to a connection  $B_0$  in a reducible bundle  $E$  over  $X_0$ , and  $B_0$  can be connected by a path to a reducible connection  $R$  which comes from a splitting  $E \cong L_0 \oplus \bar{L}_0$ . There is a standard orientation for the determinant line of the operator  $d_R^+ \oplus d_R^*$ , and this can be followed back to give an orientation for the determinant line at  $A_0$ . This determinant line is canonically oriented because the cohomology  $H_{A_0}^*$  vanishes. We compare the two orientations, which gives a sign  $\varepsilon$ .

To determine the sign at the grafted connection  $A_0 \# A_\lambda$ , note that the same sequence of instanton additions and subtractions as above relates  $A_0 \# A_\lambda$  to  $B_0 \# A_\lambda$  which can be connected to  $R \# A_\lambda$ , a reducible connection on the bundle  $L \oplus \mathbf{R}$  over  $X$ , where the Mayer-Vietoris map  $H^2(X) \rightarrow H^2(X_0) \oplus H^2(N)$  carries  $c = c_1(L)$  to  $c_1(L_0) + c_1(\lambda)$ . Since  $R \# A_\lambda$  is reducible, there is an orientation of the determinant line, and it relates to the orientation which can be pulled back from the trivial connection by  $(-1)^{c \cdot c}$ . Thus pulling the orientation back over  $A_0 \# A_\lambda$  gives the sign  $\varepsilon \cdot (-1)^{c \cdot c}$ .

To get the sign for  $A_0 \# \Theta$  we first pass to  $\mathrm{SO}(3)$ , and then  $\mathrm{ad}(A_0)$  is related as above to the reducible connection  $\mathrm{ad}(R)$  which lives in the line bundle  $L_0^2$ . Grafting to the trivial connection  $\Theta_N$ , we get  $\mathrm{ad}(A_0) \# \Theta_N$  which is connected to the reducible connection  $\mathrm{ad}(R) \# \Theta_N$ . This lives in the grafted line bundle  $L_0^2 \# \mathbf{R}$  which has  $c_1 = 2c_1(L_0)$ . (Note that although  $c_1(L_0)$  is not a global class,  $2c_1(L_0)$  is.) The class  $2c_1(L_0)$  restricts trivially to  $X_0$  and to  $N \pmod{2}$ ; so its mod 2 reduction is the same as that of  $\tau$ . (We are here identifying  $\tau$  and its Poincaré dual.) Since  $\tau = 2c_1(\lambda)$ , the difference in these reductions is  $2c_1(L_0) - \tau = 2(c - \tau)$ . The corresponding orientations compare via the parity of  $(c - \tau) \cdot (c - \tau) \equiv c \cdot c \pmod{2}$ . Thus the sign which is attached to  $A_0 \# \Theta$

is  $\varepsilon \cdot (-1)^{c \cdot c}$ , the same as for  $A_0 \# A_\lambda$ , and the sign in the formula above is ‘+’.  $\square$

For the case of the  $\mathrm{SO}(3)$  invariants the proof of Theorem 2.1 can be easily adapted to show:

**THEOREM 2.2.** *Suppose that  $\tau \in H_2(X; \mathbf{Z})$  with  $\tau \cdot \tau = -2$  is represented by an embedded sphere  $S$ . Let  $c \in H_2(X; \mathbf{Z})$  satisfy  $c \cdot \tau \equiv 0 \pmod{2}$ . Then for  $z \in \mathbf{A}(\tau^\perp)$  we have  $D_c(\tau^2 z) = 2 D_{c+\tau}(z)$ .*

We next need to review some elementary facts concerning the Donaldson invariants of blowups. These can be found, for example, in [7], [8], [12]. Let  $X$  have the Donaldson invariant  $D$ , and let  $\hat{X} = X \# \overline{\mathbf{CP}}^2$  have the invariant  $\hat{D}$ .

**LEMMA 2.3.** *Let  $e \in H_2(\overline{\mathbf{CP}}^2; \mathbf{Z}) \subset H_2(\hat{X}; \mathbf{Z})$  be the exceptional class, and let  $c \in H_2(X; \mathbf{Z})$ . Then for all  $z \in \mathbf{A}(X)$ :*

$$(1) \quad \hat{D}_c(e^{2k+1} z) = 0 \text{ for all } k \geq 0.$$

$$(2) \quad \hat{D}_c(z) = D_c(z).$$

$$(3) \quad \hat{D}_c(e^2 z) = 0.$$

$$(4) \quad \hat{D}_c(e^4 z) = -2 D_c(z).$$

$$(5) \quad \hat{D}_{c+e}(e^{2k} z) = 0 \text{ for all } k \geq 0.$$

$$(6) \quad \hat{D}_{c+e}(e z) = D_c(z).$$

$$(7) \quad \hat{D}_{c+e}(e^3 z) = -D_c(x z).$$

*Proof.* Items (1)–(5) are standard and are explained in [7]. Both (1) and (5) follow because the automorphism of  $H_2(X \# \overline{\mathbf{CP}}^2; \mathbf{Z})$  given by reflection in  $e$  is realized by a diffeomorphism. Items (2) and (3) follow from counting arguments, and (4) follows from simple arguments, as in the proof of Theorem 2.1 above [7]. Item (6) is due to D. Kotschick [8].

A proof of (7) is given in [12]. We sketch a proof here. Consider a neighborhood  $N$  of the exceptional curve, and let  $X_0 = \hat{X} \setminus N \cong X \setminus B^4$ . As in the proof of Ruberman’s theorem, we lose no generality by assuming that we are evaluating  $\hat{D}$  only on  $e^3$ . A dimension counting argument shows that if we stretch the neck between  $X_0$  and  $N$  to have infinite length by taking a sequence of generic metrics  $\{g_n\}$ , and if  $V_i$  are transverse divisors representing  $\mu(e)$ , then any sequence of connections

$$A_n \in \mathcal{M}_{\hat{X}, c+e}(g_n) \cap V_1 \cap V_2 \cap V_3$$

must converge to the sum of a connection in the 4-manifold  $\mathcal{M}_{X_0,c}$  and the unique reducible connection on  $N$  corresponding to the line bundle  $\lambda$  over  $N$  whose Euler class is Poincaré dual to  $e$ . The based moduli space  $\tilde{\mathcal{M}}_N(\lambda)$  is the orbit of this reducible connection, a 2-sphere,  $S_e^2$ . Let  $v$  denote the (positive) generator of the equivariant cohomology  $H_{\mathrm{SO}(3)}^*(S_e^2) \cong H^*(\mathbf{CP}^\infty)$  in dimension 2. The class  $\mu(e)$  lifts to the equivariant class  $-\frac{1}{2}\langle c_1(\lambda), e \rangle v = \frac{1}{2}v \in H_{\mathrm{SO}(3)}^2(S_e^2)$ . The connections in  $S_e^2$  are asymptotically trivial and this induces an  $\mathrm{SO}(3)$  equivariant push-forward map

$$\partial_*(N): H_{\mathrm{SO}(3)}^*(S_e^2) \rightarrow H_{\mathrm{SO}(3)}^*(1) = H^*(\mathrm{BSO}(3)).$$

If  $u \in H_{\mathrm{SO}(3)}^*(1)$  is the generator in dimension 4, then  $\partial_*(N)(v^{2k+1}) = 2u^k$ . Thus  $\partial_*(N)(e^3) = \frac{1}{4}u$ .

Since each connection in  $\mathcal{M}_{X_0,c}$  is also asymptotically trivial, there is an induced map  $\partial^*(X_0): H_{\mathrm{SO}(3)}^*(1) \rightarrow H_{\mathrm{SO}(3)}^*(\tilde{\mathcal{M}}_{X_0,c})$ . It follows from [16], [2] that  $\hat{D}_{c+e}(e^3)$  is obtained by evaluating

$$\begin{aligned} \frac{1}{8} \langle \partial^*(X_0) \partial_*(N)(v^3), [\tilde{\mathcal{M}}_{X_0,c}] \rangle &= \frac{1}{4} \langle \partial^*(X_0)(u), [\tilde{\mathcal{M}}_{X_0,c}] \rangle \\ &= \frac{1}{4} \langle \pi_* \partial^*(X_0)(u), [\mathcal{M}_{X_0,c}] \rangle, \end{aligned}$$

where basepoint fibration  $\beta$  over  $X_0$  is

$$\pi: \tilde{\mathcal{M}}_{X_0,c} \rightarrow \mathcal{M}_{X_0,c},$$

the last equality, because the  $\mathrm{SO}(3)$  action on  $\tilde{\mathcal{M}}_{X_0,c}$  is free. But Austin and Braam [2], for example, show that  $\pi_* \partial^*(X_0)(u) = p_1(\beta)$ . Since  $\nu = -\frac{1}{4}p_1(\beta)$ , we get  $\langle \nu, [\mathcal{M}_{X_0,c}] \rangle = -D_c(x)$ .  $\square$

We next consider embedded 2-spheres of self-intersection  $-3$ .

**THEOREM 2.4.** *Suppose that  $\tau \in H_2(X; \mathbf{Z})$  is represented by an embedded 2-sphere  $S$  with self-intersection  $-3$ . Let  $\omega \in H_2(X; \mathbf{Z})$  satisfy  $\omega \cdot \tau \equiv 0 \pmod{2}$ . Then for all  $z \in \mathbf{A}(\tau^\perp)$*

$$D_\omega(\tau z) = -D_{\omega+\tau}(z).$$

*Proof.* The proof is similar in structure to that of Theorem 2.1. Write  $X = X_0 \cup N$  where  $N$  is a tubular neighborhood of  $S$ . Then  $\partial N = L(3, -1)$ . The character variety of  $\mathrm{SO}(3)$  representations of  $\pi_1(\partial N) = \mathbf{Z}_3$  may be identified with  $H^2(\partial N; \mathbf{Z})/\{\pm 1\} = \{\eta^m \mid m \in \mathbf{Z}_3/\{\pm 1\}\}$ . Reducible anti-self-dual  $\mathrm{SO}(3)$  connections on  $N$  arise from complex line bundles  $\lambda^m$ ,  $m \in \mathbf{Z}$ , where  $\langle c_1(\lambda), \tau \rangle = -1$ . The corresponding moduli spaces  $\mathcal{M}_N(\lambda^m \oplus \mathbf{R})$  have boundary values  $\eta^m$  and dimensions  $\frac{2}{3}m^2 - 3$  if  $m \equiv 0 \pmod{3}$  and  $\frac{2m^2+1}{3} - 2$  if  $m \not\equiv 0 \pmod{3}$ .



Since it is easiest to work with an  $\omega$  which satisfies  $\partial\omega_{X_0} = 0$  (where  $\partial\omega_{X_0} \in H_1(\partial X; \mathbf{Z}) = \mathbf{Z}_3$ ), we simply work with  $\rho = 3\omega$  rather than  $\omega$ . This is no problem, since  $D_{3\omega} = (-1)^{\omega \cdot \omega} D_\omega$ . Thus we may write  $\rho = \rho_0 + \rho_N \in H_2(X_0; \mathbf{Z}) \oplus H_2(N; \mathbf{Z})$ . As in our previous arguments, we assume that we are evaluating  $D_\rho$  only on  $\tau$ . A dimension counting argument shows that  $D_\rho(\tau)$  is the product of relative invariants  $D_{X_0}[\eta^m]$  coming from a 0-dimensional cylindrical end moduli space over  $X_0$  with terms coming from nontrivial reducible connections on  $N$ . These reducible connections must live in moduli spaces of dimension  $\leq 1$ , and the corresponding line bundles must have  $c_1 \equiv \rho_N \pmod{2}$ . Our hypothesis,  $\omega \cdot \tau \equiv 0 \pmod{2}$  implies that  $\rho_N \cdot \tau \equiv 0 \pmod{2}$ ; thus the line bundle in question must be an even power of  $\lambda$ . Recalling the constraint that the dimension of the corresponding moduli space be  $\leq 1$ , the only possibility is  $\mathcal{M}_N(\lambda^2 \oplus \mathbf{R})$ .

Consider an anti-self-dual connection  $A_0$  lying in the finite 0-dimensional moduli space  $\mathcal{M}_{X_0}[\eta^2]$ , and let  $A_{\lambda^2}$  be the reducible anti-self-dual connection on  $N$ . A neighborhood of the  $\mathrm{SO}(3)$  orbit of  $A_{\lambda^2}$  in the based moduli space  $\tilde{\mathcal{M}}_N(\lambda^2 \oplus \mathbf{R})$  is modelled by  $\mathrm{SO}(3) \times_{S^1} \mathbf{C}$  and the (based) divisor for  $\tau$  is  $-\frac{1}{2}\langle c_1(\lambda^2), \tau \rangle (\mathrm{SO}(3) \times_{S^1} \{0\}) = \mathrm{SO}(3) \times_{S^1} \{0\}$ . The based connections obtained from grafting the orbit  $\mathrm{SO}(3)_{A_0}$  of  $A_0$  to the orbit of  $A_{\lambda^2}$  are given by the fibered product of these orbits over the 2-sphere in  $\mathrm{SO}(3)$  consisting of representations of  $\pi_1(\partial N)$  which are in the conjugacy class  $\eta^2$ . By cutting this down by the divisor for  $\tau$ , we obtain (up to sign) the fibered product of  $\mathrm{SO}(3)_{A_0}$  with  $S^2$  over  $S^2$ , i.e. simply  $\mathrm{SO}(3)_{A_0}$ . Taking the quotient by  $\mathrm{SO}(3)$ ,

$$D_\rho(\tau) = \pm D_{X_0}[\eta^2].$$

Since  $\eta^2 = \eta$  in the character variety, we can graft anti-self-dual connections  $A_0$  to the unique (reducible) connection  $A_\lambda$  lying in the moduli space  $\mathcal{M}_N(\lambda \oplus \mathbf{R})$  of formal dimension  $-1$ . As the glued-together bundle has  $w_2$  which is Poincaré dual to  $\rho + \tau \pmod{2}$ , we have

$$D_{\rho_0}[\eta^2] = \pm D_{\rho+\tau};$$

thus our result is proved up to a sign.

To get this sign, we need to compare signs induced at  $A_0 \# A_{\lambda^2}$  and  $A_0 \# A_\lambda$  using a fixed homology orientation of  $X$  and the integral lifts  $\rho$  and  $\rho + \tau$  of the corresponding Stiefel-Whitney classes. By an excision argument [5], the difference in signs depends only on the part of the connections over the neighborhood  $N$ . Thus the sign is universal, and may be determined by an example. For this, let  $X$  be the  $K3$  surface and  $\hat{X} = X \# \overline{\mathbf{CP}}^2$ . Let  $s$  be any class in  $H_2(X)$  of square  $-2$  represented by an embedded 2-sphere (e.g. a section), and let  $\tau = s + e$ . Note that  $s - 2e \in \mathbf{A}(\tau^\perp)$ . Then using Theorems

2.1 and 2.3,

$$\begin{aligned}\hat{D}((s-2e)\tau) &= D(s^2) = 2D_s \\ \hat{D}_\tau(s-2e) &= -2D_s;\end{aligned}$$

so the overall sign is ‘ $-$ ’.

□

Next we combine our two relations to obtain a relation which is crucial in obtaining the general blowup formula. This relation was first proved by Wojciech Wieczorek using different methods. His proof appears in his thesis [17].

**COROLLARY 1.** *Suppose that  $\tau \in H_2(X; \mathbf{Z})$  is represented by an embedded 2-sphere with self-intersection  $-2$ , and let  $c \in H_2(X; \mathbf{Z})$  with  $c \cdot \tau \equiv 0 \pmod{2}$ . Then for all  $z \in \mathbf{A}(\tau^\perp)$*

$$D_c(\tau^4 z) = -4D_c(\tau^2 x z) - 4D_c(z).$$

*Proof.* In  $\hat{X} = X \# \overline{\mathbf{CP}}^2$  the class  $\tau + e$  is represented by a 2-sphere of self-intersection  $-3$ , and  $(\tau - 2e) \cdot (\tau + e) = 0$ . From Lemma 2.3,

$$\hat{D}_c((\tau - 2e)^3(\tau + e)z) = D_c(\tau^4 z) - 8\hat{D}_c(e^4 z) = D_c(\tau^4 z) + 16D_c(z).$$

On the other hand, by Theorems 2.2 and 2.4 and by Lemma 2.3,

$$\begin{aligned}\hat{D}_c((\tau - 2e)^3(\tau + e)z) &= -\hat{D}_{c+\tau+e}((\tau - 2e)^3 z) \\ &= 6\hat{D}_{c+\tau+e}(\tau^2 e z) + 8\hat{D}_{c+\tau+e}(e^3 z) \\ &= 6D_{c+\tau}(\tau^2 z) - 8D_{c+\tau}(x z) \\ &= 12D_c(z) - 4D_c(\tau^2 x z),\end{aligned}$$

and the result follows.

□

### 3. The blowup equation

Let  $X$  be a simply connected oriented 4-manifold and let  $\hat{X} = X \# \overline{\mathbf{CP}}^2$ . Let  $c \in H_2(X; \mathbf{Z})$ . Of course  $H_2(\hat{X}; \mathbf{Z}) = H_2(X; \mathbf{Z}) \oplus \mathbf{Z}e$  with  $e$  the exceptional class. It follows from Lemma 2.3(1),(5) that we can write

$$\hat{D}_c = \sum \beta_{c,k} E^{2k}$$

where  $E$  denotes the 1-form given by  $E(y) = e \cdot y$  and  $\beta_{c,k}(\alpha^d) = \hat{D}_c(\alpha^d e^{2k})$  for any  $\alpha \in H_2(X)$ . Similarly

$$\hat{D}_{c+e} = \sum \gamma_{c,k} E^{2k+1}.$$

Consider  $\bar{X} = X \# 2\overline{\mathbf{CP}}^2$  with exceptional classes  $e_1, e_2 \in H_2(\bar{X}; \mathbf{Z})$ , and let  $\bar{D}$  denote its Donaldson invariant. Then  $e_1 - e_2$  has self-intersection  $-2$  and is represented by an embedded 2-sphere. Furthermore, the intersection  $(e_1 - e_2) \cdot (e_1 + e_2) = 0$ ; thus we can apply Corollary 2.4 to get

$$(3.1) \quad \begin{aligned} \bar{D}_c((e_1 + e_2)^r (e_1 - e_2)^4 z) \\ = -4 \bar{D}_c((e_1 + e_2)^r (e_1 - e_2)^2 x z) - 4 \bar{D}_c((e_1 + e_2)^r z) \end{aligned}$$

for all  $z \in \mathbf{A}(X)$ .

LEMMA 3.1. *There are polynomials,  $B_k(x)$ , independent of  $X$ , so that for any  $c \in H_2(X; \mathbf{Z})$  and  $z \in \mathbf{A}(X)$  we have  $\hat{D}_c(e^k z) = D_c(B_k(x) z)$ .*

*Proof.* Lemma 2.3(2) implies that  $\beta_{c,0} = D_c$ . Thus we have  $B_0 = 1$ . Furthermore, 2.3(1) implies that  $B_{2k+1}(x) = 0$  for all  $k$ . Assume inductively that for  $j \leq k$ ,  $\hat{D}_c(e^j z) = D_c(B_j(x) z)$ . Expanding (3.1) via the induction hypothesis, we have

$$\begin{aligned} \bar{D}_c((e_1 + e_2)^{k-3} (e_1 - e_2)^4 z) &= -4 D_c\left(z \sum_{i=0}^{k-3} \binom{k-3}{i} \{x B_{i+2}(x) B_{k-3-i}(x) \right. \\ &\quad \left. - 2x B_{i+1}(x) B_{k-2-i}(x) + x B_i(x) B_{k-1-i}(x) + B_i(x) B_{k-3-i}(x)\} \right) = D_c(P(x) z) \end{aligned}$$

for some polynomial  $P$ . On the other hand, by expanding the argument of

$$\bar{D}_c((e_1 + e_2)^{k-3} (e_1 - e_2)^4 z),$$

and using the induction hypothesis in a similar fashion, we get

$$\bar{D}_c((e_1 + e_2)^{k-3} (e_1 - e_2)^4 z) = 2 \hat{D}_c(e^{k+1} z) + D_c(R(x) z)$$

for another polynomial  $R$ . The lemma follows.  $\square$

LEMMA 3.2. *There are polynomials,  $S_k(x)$ , independent of  $X$ , so that for any  $c \in H_2(X; \mathbf{Z})$  and  $z \in \mathbf{A}(X)$  we have  $\hat{D}_{c+e}(e^k z) = D_c(S_k(x) z)$ .*

*Proof.* By Theorem 2.1 we have for any even  $k > 0$ ,

$$\bar{D}_c((e_1 + e_2)^k (e_1 - e_2)^2) = 2 D_{c+e_1-e_2}((e_1 + e_2)^k) = -2 D_{c+e_1+e_2}((e_1 + e_2)^k).$$

This formula can then be used as above to inductively calculate  $S_{k-1}(x)$  in terms of  $S_1(x), \dots, S_{k-3}(x)$  and  $B_0(x), \dots, B_{k+2}(x)$ .  $\square$

We now explicitly determine the polynomials  $B_k(x)$  and  $S_k(x)$ . Set

$$B(x, t) = \sum_{t=0}^{\infty} B_k(x) \frac{t^k}{k!} \quad \text{and} \quad S(x, t) = \sum_{t=0}^{\infty} S_k(x) \frac{t^k}{k!}.$$

Note that

$$\begin{aligned}
 (3.2) \quad \frac{d^n}{dt^n} \hat{D}(\exp(te) z) &= \hat{D} \left( z \sum e^{k+n} \frac{t^k}{k!} \right) \\
 &= D \left( z \sum B_{k+n}(x) \frac{t^k}{k!} \right) \\
 &= \frac{d^n}{dt^n} D(B(x, t) z) = D(B^{(n)}(x, t) z)
 \end{aligned}$$

where the last differentiation is with respect to  $t$ . On  $\bar{X} = X \# 2\overline{\mathbf{CP}}^2$ , we get  $\bar{D}(\exp(t_1 e_1 + t_2 e_2) z) = D(B(x, t_1) B(x, t_2) z)$ . Now apply Corollary 2.4 to  $e_1 - e_2 \in H_2(\bar{X}; \mathbf{Z})$ . Since for any  $t \in \mathbf{R}$  the class  $te_1 + te_2 \in \langle e_1 - e_2 \rangle^\perp$ , we have the equation

$$\begin{aligned}
 (3.3) \quad &\bar{D}(\exp(te_1 + te_2) (e_1 - e_2)^4 z) \\
 &+ 4\bar{D}(x \exp(te_1 + te_2) (e_1 - e_2)^2 z) + 4\bar{D}(\exp(te_1 + te_2) z) = 0.
 \end{aligned}$$

But, for example,

$$e_1^4 \exp(te_1 + te_2) = \left( \sum e_1^{k+4} \frac{t^k}{k!} \right) \left( \sum e_2^k \frac{t^k}{k!} \right) = \frac{d^4}{dt^4} (\exp(te_1)) \exp(te_2).$$

We argue similarly and use (3.2) to obtain

$$\begin{aligned}
 &\bar{D}(\exp(te_1 + te_2) (e_1 - e_2)^4 z) \\
 &= D((2B^{(4)}(x, t) B(x, t) - 8B'''(x, t) B'(x, t) + 6(B''(x, t))^2) z) \\
 &= 2D((B^{(4)} B - 4B''' B' + 3(B'')^2) z)
 \end{aligned}$$

where  $B = B(x, t)$ . Completing the expansion of (3.3), we get

$$2D((B^{(4)} B - 4B''' B' + 3(B'')^2 + 4x(B'' B - (B')^2) + 2B^2) z) = 0$$

for all  $z \in \mathbf{A}(X)$ . This means that the expression

$$B^{(4)} B - 4B''' B' + 3(B'')^2 + 4x(B'' B - (B')^2) + 2B^2$$

lies in the kernel of  $D$ :  $\mathbf{A}(X) \rightarrow \mathbf{R}$ .

Thus the “blowup function”  $B(x, t)$  satisfies the differential equation

$$B^{(4)} B - 4B''' B' + 3(B'')^2 + 4x(B'' B - (B')^2) + 2B^2 = 0$$

modulo the kernel of  $D$ . Of course, the fact that this equation holds only modulo the kernel of  $D$  is really no constraint, since our interest in  $B(x, t)$  comes from the equation  $\hat{D}(\exp(te) z) = D(B(x, t) z)$ .

Considering  $x$  as fixed, we view  $B(x, t)$  as a function of  $t$  and let  $B = \exp(f(t))$ .

PROPOSITION 3.3. *Modulo the kernel of  $D$ , the logarithm  $f(t)$  of  $B(x, t)$  satisfies the differential equation*

$$f^{(4)} + 6(f'')^2 + 4xf'' + 2 = 0$$

*with the initial conditions  $f(0) = f'(0) = f''(0) = f'''(0) = 0$ .*

The initial conditions follow from Lemma 2.3.

#### 4. The blowup formula

In order to solve the differential equation of Proposition 3.3, we set  $u = f''$ . Then the differential equation becomes

$$u'' + 6u^2 + 4xu + 2 = 0$$

with initial conditions  $u(0) = u'(0) = 0$ . This is equivalent to the equation

$$(4.1) \quad (u')^2 = -4u^3 - 4xu^2 - 4u,$$

as can be seen by differentiating the last equation with respect to  $t$ . We replace  $u$  by  $-v$  and complete the cube which yields

$$(v')^2 = 4\left(v - \frac{x}{3}\right)^3 - \frac{4}{3}vx^2 + \frac{4x^3}{27} + 4v.$$

Finally, letting  $y = v - \frac{x}{3}$  we get

$$(4.2) \quad (y')^2 = 4y^3 - g_2y - g_3 \quad \text{where} \quad g_2 = 4\left(\frac{x^2}{3} - 1\right) \quad \text{and} \quad g_3 = \frac{8x^3 - 36x}{27}.$$

This is the equation which defines the Weierstrass  $\wp$ -function. In fact, if we rewrite (4.2) as

$$\frac{dt}{dy} = \frac{1}{\sqrt{4y^3 - g_2y - g_3}},$$

then

$$t = \int_y^\infty \frac{ds}{\sqrt{4s^3 - g_2s - g_3}} = \wp^{-1}(y)$$

and we see that for arbitrary constants  $c$ ,  $y = \wp(t + c)$  gives all solutions to (4.2), and so  $f'' = u = -(\wp(t + c) + \frac{x}{3})$  is the general solution of (4.1).

The roots of the cubic equation

$$4s^3 - g_2s - g_3 = 0$$

are

$$(4.3) \quad e_1 = \frac{x}{6} + \frac{\sqrt{x^2 - 4}}{2}, \quad e_2 = \frac{x}{6} - \frac{\sqrt{x^2 - 4}}{2}, \quad e_3 = -\frac{x}{3}$$

where we have followed standard notation (cf. [1]). These correspond to the half-periods  $\omega_i = \wp^{-1}(e_i)$  of the  $\wp$ -function. The initial condition  $f''(0) = 0$  implies that  $\wp(c) = -\frac{x}{3} = e_3$ ; so  $c = \omega_3 + 2\varpi$ , where  $2\varpi = 2m_1\omega_1 + 2m_3\omega_3$ , with  $m_1, m_3 \in \mathbf{Z}$ , is an arbitrary period. (Note that the initial condition  $f'''(0) = 0$  follows because the half-periods are zeros of  $\wp'$ .) The Weierstrass zeta-function satisfies  $\zeta' = -\wp$ ; thus  $f'(t) = \zeta(t + \omega_3 + 2\varpi) - \frac{tx}{3} + a$ . The constant  $a$  is determined by the initial condition  $f'(0) = 0$ ;  $a = -\zeta(\omega_3 + 2\varpi)$ . Since the logarithmic derivative of the Weierstrass sigma-function is  $\zeta$ , integrating one more time gives  $f(t) = \log \sigma(t + \omega_3 + 2\varpi) - t\zeta(\omega_3 + 2\varpi) - \frac{t^2x}{6} + b$ , and the initial condition  $f(0) = 0$  shows that  $b = -\log \sigma(\omega_3 + 2\varpi)$ . Thus

$$B(x, t) = e^{f(t)} = e^{-\frac{t^2x}{6}} e^{-t\zeta(\omega_3 + 2\varpi)} \frac{\sigma(t + \omega_3 + 2\varpi)}{\sigma(\omega_3 + 2\varpi)}.$$

For  $\omega = \omega_1$  or  $\omega_3$  and  $\eta = \zeta(\omega)$  we have the formulas

$$\zeta(u + 2m\omega) = 2m\eta + \zeta(u), \quad \sigma(u + 2m\omega) = (-1)^m e^{2\eta(mu + m^2\omega)} \sigma(u)$$

(which follow easily from [1, p. 199]). Using them, our formula for  $B(x, t)$  becomes

$$B(x, t) = e^{-\frac{t^2x}{6}} e^{-\eta_3 t} \frac{\sigma(t + \omega_3)}{\sigma(\omega_3)}.$$

The above addition formula for the sigma-function implies that

$$\sigma(t + \omega_3) = \sigma((t - \omega_3) + 2\omega_3) = -e^{2\eta_3 t} \sigma(t - \omega_3).$$

Thus

$$B(x, t) = -e^{-\frac{t^2x}{6}} e^{\eta_3 t} \frac{\sigma(t - \omega_3)}{\sigma(\omega_3)} = e^{-\frac{t^2x}{6}} \sigma_3(t),$$

the last equality by the definition of the quasi-periodic function  $\sigma_3$ . In conclusion, recording the fact that  $\sigma_3$  also depends upon  $x$ , we have:

**THEOREM 4.1.** *Modulo the kernel of  $D$ , the blowup function  $B(x, t)$  is given by the formula*

$$B(x, t) = e^{-\frac{t^2x}{6}} \sigma_3(x, t);$$

i.e., for any  $c \in H_2(X; \mathbf{Z})$  and  $z \in \mathbf{A}(X)$ ,

$$\hat{D}_c(\exp(te) z) = D_c(e^{-\frac{t^2x}{6}} \sigma_3(x, t) z).$$

The indexing of the Weierstrass functions  $\sigma_i$  depends on the ordering of the roots  $e_i$  of the equation  $4s^3 - g_2s - g_3 = 0$ . This can be confusing. The important point is that the sigma-function we are using corresponds to the

root  $-\frac{x}{3}$ . One can now obtain the individual blowup polynomials from the formula for  $B(x, t)$ . For example,  $B_{12} = -512x^4 - 960x^2 - 408$  and (for fun),

$$\begin{aligned} B_{30}(x) &= 134,217,728x^{13} + 4,630,511,616x^{11} + 68,167,925,760x^9 \\ &\quad - 34,608,135,536,640x^7 - 39,641,047,695,360x^5 \\ &\quad - 9,886,101,110,784x^3 + 543,185,367,552x. \end{aligned}$$

(We thank Alex Selby for help with some computer calculations.) We also obtain:

**THEOREM 4.2.** *Modulo the kernel of  $D$ , the blowup function  $S(x, t)$  is given by the formula*

$$S(x, t) = e^{-\frac{t^2x}{6}} \sigma(x, t);$$

i.e., for any  $c \in H_2(X; \mathbf{Z})$  and  $z \in \mathbf{A}(X)$ ,

$$\hat{D}_{c+e}(\exp(te)z) = D_c(e^{-\frac{t^2x}{6}} \sigma_3(x, t)z).$$

*Proof.* As usual we let  $\bar{X} = X \# 2\overline{\mathbf{CP}}^2$  with exceptional classes  $\varepsilon_1$  and  $\varepsilon_2$ . (We have temporarily changed notation to avoid confusion with the roots  $e_i$  of  $4s^3 - g_2s - g_3 = 0$ .) Consider the class  $\varepsilon_1 - \varepsilon_2$  which is represented by a sphere of self-intersection  $-2$ . By Theorem 2.1,

$$\begin{aligned} \bar{D}(\exp(t\varepsilon_1 + t\varepsilon_2)(\varepsilon_1 - \varepsilon_2)^2 z) &= 2D_{\varepsilon_1 - \varepsilon_2}(\exp(t\varepsilon_1 + t\varepsilon_2)z) \\ &= -2D_{\varepsilon_1 + \varepsilon_2}(\exp(t\varepsilon_1 + t\varepsilon_2)z) \end{aligned}$$

for all  $z \in \mathbf{A}(X)$ . Equivalently we get  $D((2B''B - 2(B')^2)z) = -2D(S^2z)$ . In other words,

$$S^2 = e^{-\frac{t^2x}{3}} \left( \frac{x}{3} \sigma_3^2 + (\sigma_3')^2 - \sigma_3 \sigma_3'' \right).$$

Write  $\sigma_3(t) = \exp(h(t))$ . Then

$$S^2 = e^{-\frac{t^2x}{3}} e^{2h} \left( \frac{x}{3} - h'' \right);$$

i.e.,

$$S = \pm e^{-\frac{t^2x}{6}} \sigma_3(t) \left( \frac{x}{3} - h'' \right)^{\frac{1}{2}}.$$

Since  $\exp(h) = \sigma_3(t) = \sigma(t)(\wp(t) - e_3)^{\frac{1}{2}}$ , it follows that  $h = \log \sigma(t) + \frac{1}{2} \log(\wp(t) - e_3)$ . Then

$$h' = \zeta(t) + \frac{1}{2} \frac{\wp'(t)}{\wp(t) - e_3} = \zeta(t) + \frac{1}{2} (\zeta(t + \omega_3) + \zeta(t - \omega_3) - 2\zeta(t))$$

by [1, p. 41]. Thus  $h' = \frac{1}{2}(\zeta(t + \omega_3) + \zeta(t - \omega_3))$ , and

$$h'' = \frac{1}{2}(-\wp(t + \omega_3) - \wp(t - \omega_3)) = -\wp(t + \omega_3).$$

Also

$$S = \pm e^{-\frac{t^2 x}{6}} \sigma_3(t) (\wp(t + \omega_3) - e_3)^{\frac{1}{2}} = \pm e^{-\frac{t^2 x}{6}} \sigma_3(t) \left( \frac{(e_3 - e_1)(e_3 - e_2)}{\wp(t) - e_3} \right)^{\frac{1}{2}}$$

([1, p. 200]). However,  $(e_3 - e_1)(e_3 - e_2) = 1$  (see (4.3)); so

$$S = \pm e^{-\frac{t^2 x}{6}} \frac{\sigma_3(t)}{\sqrt{\wp(t) - e_3}} = \pm e^{-\frac{t^2 x}{6}} \sigma(t).$$

To determine the sign, note that (fixing  $x$ )  $S'(0) = S_1 = 1$  by Theorem 2.3(6). But from our formula  $S'(0) = \pm \sigma'(0)$ , and  $\sigma'(0) = 1$ .  $\square$

## 5. The blowup formula for manifolds of simple type

A 4-manifold is said to be of *simple type* [10] if for all  $z \in \mathbf{A}(X)$  the relation  $D(x^2 z) = 4D(z)$  is satisfied by its Donaldson invariant. It is clear that if  $X$  has simple type, then  $\hat{X} = X \# \overline{\mathbf{CP}}^2$  does as well. In this case, following [10], one considers the invariant  $\mathbf{D}$  defined by

$$\mathbf{D}(\alpha) = D\left(\left(1 + \frac{x}{2}\right) \exp(\alpha)\right)$$

for all  $\alpha \in H_2(X)$ .  $\mathbf{D}$  is called the *Donaldson series* of  $X$ . Note that the simple type condition implies that for any  $z \in \mathbf{A}(X)$ ,

$$D\left(\left(1 + \frac{x}{2}\right) z\right) = 2D\left(\left(1 + \frac{x}{2}\right) z\right);$$

i.e.,  $x$  acts as multiplication by 2 on  $D(1 + \frac{x}{2})$ . The blowup formula in this case has been determined previously by Kronheimer and Mrowka. In this section, we derive that formula by setting  $x = 2$  in Theorems 4.1 and 4.2. This gives a degenerate case of the associated Weierstrass functions. All the formulas below involving elliptic functions can be found in [1]. The squares  $k^2$ ,  $k'^2$  of the modulus and complementary modulus of our Weierstrass functions are given by

$$k^2 = \frac{x - \sqrt{x^2 - 4}}{x + \sqrt{x^2 - 4}} \quad k'^2 = \frac{2\sqrt{x^2 - 4}}{x + \sqrt{x^2 - 4}}.$$

Thus  $k^2 = 1$  and  $k'^2 = 0$  when  $x = 2$ . The corresponding complete elliptic



integrals of the first kind are

$$K = \int_0^1 \frac{ds}{\sqrt{(1-s^2)(1-k^2s^2)}} = \int_0^1 \frac{ds}{1-s^2}$$

$$K' = \int_0^1 \frac{ds}{\sqrt{(1-s^2)(1-k'^2s^2)}} = \int_0^1 \frac{ds}{\sqrt{1-s^2}}.$$

Thus  $K = \infty$  and  $K' = \frac{\pi}{2}$  when  $x = 2$ . Also, when  $x = 2$  we have  $g_2 = \frac{4}{3}$  and  $g_3 = -\frac{8}{27}$ ; hence the roots of  $4s^3 - g_2s - g_3 = 0$  are  $e_1 = e_2 = \frac{1}{3}$  and  $e_3 = -\frac{2}{3}$ . This means that when  $x = 2$  the basic periods are

$$\omega_1 = \frac{K}{\sqrt{e_1 - e_3}} = K = \infty \quad \omega_3 = \frac{iK'}{\sqrt{e_1 - e_3}} = iK' = \frac{i\pi}{2}.$$

In this situation,

$$\sigma(t) = \frac{2\omega_3}{\pi} e^{\frac{1}{6}(\frac{\pi t}{2\omega_3})^2} \sin \frac{\pi t}{2\omega_3} = e^{-\frac{t^2}{6}} \sinh t,$$

and

$$\wp(t) = -\frac{1}{3} \left( \frac{\pi}{2\omega_3} \right)^2 + \left( \frac{\pi}{2\omega_3} \right)^2 \frac{1}{\sin^2(\frac{\pi t}{2\omega_3})} = \frac{1}{3} + \frac{1}{\sinh^2 t}.$$

Thus

$$\sigma_3(t) = \sigma(t) \sqrt{\wp(t) - e_3} = e^{-\frac{t^2}{6}} \sinh t \sqrt{1 + \frac{1}{\sinh^2 t}} = e^{-\frac{t^2}{6}} \cosh t.$$

**THEOREM 5.1.** *If  $X$  has simple type, the Donaldson series of  $\hat{X} = X \# \overline{\mathbf{CP}}^2$  is*

$$\hat{\mathbf{D}} = \mathbf{D} \cdot e^{-\frac{E^2}{2}} \cosh E$$

where  $E$  is the form dual to the exceptional class  $e$ , i.e.  $E(z) = e \cdot z$  for all  $z \in H_2(\hat{X})$ . Also

$$\hat{\mathbf{D}}_e = -\mathbf{D} \cdot e^{-\frac{E^2}{2}} \sinh E.$$

*Proof.* For  $\alpha \in H_2(X)$  we calculate

$$\begin{aligned} \hat{\mathbf{D}}(\alpha + te) &= \hat{D} \left( \left( 1 + \frac{x}{2} \right) \exp(\alpha) \exp(te) \right) = D \left( \left( 1 + \frac{x}{2} \right) \exp(\alpha) B(x, t) \right) \\ &= D \left( \left( 1 + \frac{x}{2} \right) \exp(\alpha) e^{-\frac{t^2 x}{6}} e^{-\frac{t^2}{6}} \cosh t \right). \end{aligned}$$

The simple type condition implies that  $D((1 + \frac{x}{2})e^{-\frac{t^2 x}{6}}) = D((1 + \frac{x}{2})e^{-\frac{t^2}{3}})$ .

Hence

$$\hat{\mathbf{D}}(\alpha + te) = \mathbf{D}(\alpha)e^{-\frac{t^2}{2}} \cosh t = \mathbf{D}(\alpha)(e^{-\frac{E^2}{2}} \cosh E)(te)$$

as desired. The formula for  $\hat{\mathbf{D}}_e$  follows similarly since  $\sinh(E)(te) = -\sinh(t)$ .  $\square$

It is shown in [6], and also by Kronheimer and Mrowka [11], that if  $X$  has simple type, then for all  $c \in H_2(X; \mathbf{Z})$  we have  $D_c(x^2 z) = 4 D_c(z)$  for all  $z \in \mathbf{A}(X)$ . As above, we obtain:

THEOREM 5.2. *If  $X$  has simple type, then*

$$\begin{aligned}\hat{\mathbf{D}}_c &= \mathbf{D}_c \cdot e^{-\frac{E^2}{2}} \cosh E. \\ \hat{\mathbf{D}}_{c+e} &= -\mathbf{D}_c \cdot e^{-\frac{E^2}{2}} \sinh E.\end{aligned}$$

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