Computing Donaldson Invariants

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Abstract. In these lectures we will discuss techniques for constructing simply-connected smooth 4-manifolds and computing their Donaldson invariants. There are 37 Exercises (which can be solved by the diligent reader) and 24 Problems whose solutions should advance our understanding of smooth 4-manifolds.[4] Most of the material in these lectures is either drawn from (or is the basis of) other work by the author and Ronald A. Fintushel.

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[4]Added in proof: These lectures were delivered in July 1994. A few months later, gauge theory underwent the Seiberg-Witten revolution. Many of the results presented in these lectures now have more easily stated and proved counterparts in this new Seiberg-Witten theory. However, of the 24 Problems presented in these lectures, only Problems 5, 7, 8, 11, 12, 16 have either partial or complete answers. These new developments (up to June 1997) will appear as footnotes.

LECTURE 1
Overview

1.1. Classical invariants

Our problem is to classify closed smooth 4-manifolds. To avoid the group theoretic problems arising from the fact that any finitely presented group can occur as the fundamental group of a smooth closed 4-manifold, we assume our manifolds are simply-connected. Most of the classical invariants for 4-manifolds are encoded by the intersection form Q_X. This form is an integral unimodular symmetric bilinear pairing

$$Q_X : H_2(X;\mathbf{Z}) \otimes H_2(X;\mathbf{Z}) \to \mathbf{Z}$$

obtained by representing homology classes as oriented embedded submanifolds and counting intersections with signs; it is Poincaré dual to the pairing given by cup product. From the intersection form one can determine its **rank** (which is the second Betti number $b_2(X) = \mathrm{rank} H_2(X;\mathbf{Z}) = b^+(X) + b^-(X)$), its **signature** $= \sigma(X) = b^+(X) - b^-(X)$ (where $b^\pm(X)$ are the dimensions of the $\pm$-eigenspaces of Q_X) and its **type** (which is **even** if $Q_X(x,x) \equiv 0 \mod 2$ for all x, and **odd** otherwise). A form is **definite** provided one of $b^\pm(X)$ vanishes, and is **indefinite** otherwise. Note that the Euler characteristic $e(X)$ of X is $b^+(X)+b^-(X)+2$. Also, a simply-connected 4-manifold with even intersection form is a spin 4-manifold (i.e. $w_2(X) = 0$). Further, $Q_{X\#Y} = Q_X \oplus Q_Y$. Given a basis for $H_2(X;\mathbf{Z})$ we can represent Q_X as a symmetric integral matrix with determinant equal to ± 1.

The standard examples of 4-manifolds and their intersection forms are:

$$\begin{array}{ll} S^4 & Q_{S^4} = \emptyset \\ \mathrm{CP}^2 & Q_{\mathrm{CP}^2} = (1) \\ S^2 \times S^2 & Q_{S^2\times S^2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = H \\ \mathrm{K3} & Q_{K3} = E_8 \oplus E_8 \oplus 3H \end{array}$$

Here $K3$ is the Kummer or K3 surface which is diffeomorphic to the quartic surface in CP^3 and E_8 is the intersection form for the E_8 plumbing manifold:

1.1.7. ***Uniqueness.*** *Given a closed, simply-connected smooth 4-manifold M, determine the distinct smooth structures on M.*

These questions are the subject of active current research.

1.2. Existence

The early work of Donaldson and its derivatives ([**D1**],[**D3**],[**FS1**],[**FS2**]) has shown that there are indeed restrictions on the intersection form of a closed, simply-connected 4-manifold and has made progress towards a verification of the following existence conjecture:

1.2.1. ***11/8 Conjecture.*** *The intersection form of a closed, simply-connected smooth 4-manifold must be either*

- *diagonalizable (over $\mathbf{Z}$), or*
- *even and $\frac{b_2(X)}{|\sigma(X)|} \geq \frac{11}{8}$.*

Exercise 1. Complex surfaces and their connected sums satisfy the 11/8 Conjecture.

All diagonalizable forms are realized by the connected sums of $\pm \mathbf{C}P^2$. Although the classification of definite integral forms is an active area of research and far from an accomplished feat, nature has a way of taking care of all this. Donaldson [**D1**] first showed that no non-diagonalizable definite form can be realized as the intersection form of a closed smooth 4-manifold; shortly thereafter he [**D3**] showed that if the form $nE_8 \oplus mH$, $n \neq 0$, is realized, then $m \geq 3$. The $K3$-surface realizes $m = 3$.

There has not been much progress on the 11/8 Conjecture beyond the original work of Donaldson. One approach to these existence questions amounts to constructing 4-manifolds with a given Euler characteristic, signature, and type; i.e. it is a geography question. For a long time the only manifolds around were complex manifolds and their connected sums. A long standing conjecture held that this was all there was; i.e., every smooth simply connected closed 4-manifold is the connected sum of manifolds, each admitting a complex structure with one of its orientations. (For this conjecture, the 4-sphere was considered as the empty connected sum.) In 1990 this was disproved by Gompf and Mrowka [**GM**] who produced infinite families of examples of 4-manifolds homeomorphic to the K3 surface and which admit no complex structure with either orientation. Then in 1991 Fintushel and I [**FS6**] constructed irreducible 4-manifolds that (with either orientation) were not homeomorphic to any complex surface. Here, by an *irreducible* 4-manifold we mean that for each splitting as a connected sum $X_1 \# X_2$, one of the X_i is a homotopy sphere (thus avoiding the 4-dimensional Poincaré conjecture). In these lectures we will give more elementary constructions of such manifolds. This work will suggest an even stronger conjecture.

1.2.2. ***3/2 Conjecture.*** *The intersection form of an irreducible simply-connected closed smooth 4-manifold has the property that $2e(X) - 3|\sigma(X)| \geq 0$.*

Figure 1 illustrates those regions for which irreducible simply-connected closed, smooth 4-manifolds are known to exist. The co-ordinates mimic those for the

Problem 6. Find an example of a smooth simply-connected 4-manifold with $b^+ > 1$ with at most finitely many smooth structures.

1.3. Uniqueness (The Donaldson Invariant)

No reasonable Uniqueness Conjecture has emerged. (However unreasonable ones will emerge during these lectures.) Herein lies the current active work in this area. Here, we are studying smooth 4-manifolds which have the same intersection form (and hence are homeomorphic).

Exercise 6. The intersection form of a simply-connected smooth 4-manifold is completely determined by its rank, signature, and type.

Thus a new invariant is required and this begins another of Donaldson's contributions. Given an oriented simply connected 4-manifold with a generic Riemannian metric and an $SU(2)$ or $SO(3)$ bundle P over X, the moduli space of gauge equivalence classes of anti-self-dual connections on P is a manifold $\mathcal{M}_X(P)$ of dimension

$$8\,c_2(P) - 3\,(1 + b_X^+)$$

if P is an $SU(2)$ bundle, and

$$-2p_1(P) - 3\,(1 + b_X^+)$$

if P is an $SO(3)$ bundle. It will often be convenient to treat these two cases together by identifying $\mathcal{M}_X(P)$ and $\mathcal{M}_X(\mathrm{ad}(P))$ for an $SU(2)$ bundle P. Over the product $\mathcal{M}_X(P) \times X$ there is a universal $SO(3)$ bundle $\mathbf{P}$ and there results a homomorphism $\mu : H_i(X) \to H^{4-i}(\mathcal{M}_X(P))$ obtained by decomposing the class $-\frac{1}{4}p_1(\mathbf{P}) \in H^4(\mathcal{M}_X \times X)$. (Homology is always taken to have real coefficients unless it is otherwise adorned.) The basic idea of Donaldson's theory is that one should evaluate cup products of classes in the image of μ against the fundamental class of $\mathcal{M}_X(P)$. To do this, one first needs to orient $\mathcal{M}_X(P)$. This is accomplished by orienting $H^2_+(X)$ (see [**D5**]). If P is an $SO(3)$ bundle, we fix an integral lift of $w_2(P) \in H^2(X;\mathbf{Z}_2)$ and always identify such a lift with its Poincaré dual $c \in H_2(X;\mathbf{Z})$. The Pontryagin number $p_1(P)$ is congruent to c^2 (mod 4). If c and c' are two integral "lifts" of $w_2(P)$, then the difference in induced orientations is given by $(-1)^{(\frac{c-c'}{2})^2}$. We say that c and c' are equivalent if they are congruent (mod2) and if $(-1)^{(\frac{c-c'}{2})^2} = +1$. The combination of the orientations of X and $H^2_+(X)$ together with an equivalence class c of lifts of $w_2(P)$ is called a "homology orientation" of X. (In case P is an $SU(2)$ bundle, one chooses $c = 0$.) For a Kähler surface X with Kähler class K_X, there is a natural orientation induced from the Kähler structure and a choice of a lift c gives an orientation which differs from this one by $(-1)^{\frac{1}{2}(c^2+c\cdot K_X)}$ [**D5**].

The moduli space $\mathcal{M}_X(P)$ is, in general, noncompact and needs to be compactified before a fundamental class can be defined. The Uhlenbeck compactification $\overline{\mathcal{M}}_X(P)$ is well-suited to this. However, this compactification is a stratified space and is not usually a manifold. Thus, to define a fundamental class one needs to insure that the singular set has codimension at least 2. This turns out to be the case when either $w_2(P) \neq 0$ or when $w_2(P) = 0$, $d > \frac{3}{4}(1 + b_X^+)$. In practice, one is able to get around this latter restriction by blowing up X and considering bundles over $X\#\overline{\mathbf{CP}}^2$ which are nontrivial when restricted to the exceptional divisor [**MM**].

to $-c^2 + \frac{1}{2}(1 + b^+) \pmod 2$. The *Donaldson series* $\mathbf{D}_c = \mathbf{D}_{X,c}$ is defined by

$$\mathbf{D}_{X,c}(\alpha) = \hat{D}_{X,c}(\exp(\alpha)) = \sum_{d=0}^{\infty} \frac{\hat{D}_{X,c}(\alpha^d)}{d!}$$

for all $\alpha \in H_2(X)$. This is a formal power series on $H_2(X)$.

The first general properties of the Donaldson invariant were given by Donaldson **[D6]**. These include a vanishing theorem ($D_{X\#Y} \equiv 0$ if both $b^+(X) > 0$ and $b^+(Y) > 0$), and a non-vanishing theorem ($D_X(h^r) > 0$ ($r >> 0$) if X is an algebraic surface and h is an ample divisor).

We can use these theorems, together with Donaldson's first theorem (Theorem A) to determine, in certain instances, if a 4-manifold is irreducible.

Exercise 7. If X is spin and $D_X \neq 0$, then X is irreducible.

Exercise 8. If X is not spin with $D_X \neq 0$ and $X = X_1 \# X_2$, then one of the X_i has a negative definite intersection form.

Problem 7. Determine if a minimal algebraic surface is irreducible. More generally, when $X = X_1 \# X_2$, determine if one of X_1 or X_2 a connected sum of $\overline{\mathbf{CP}}^2$'s. [3]

The Donaldson invariant D_X is a refinement of the intersection form Q_X, and it too has its incumbent existence and uniqueness questions.

Problem 8. (Partially solved (cf. Theorem 1.3.1)) Which elements $D \in \mathbf{A}^*(X)$ occur as the Donaldson invariant for some smooth 4-manifold X; which formal power series on $H_2(X)$ occur as the Donaldson series of a smooth 4-manifold?

Problem 9. Are there examples of non diffeomorphic simply-connected smooth 4-manifolds X and Y with $D_X = D_Y$. Potential examples are the Horikowa surfaces which are discussed in the last lecture.

Problem 10. Determine canonical procedures for producing all simply-connected smooth 4-manifolds so that the resulting Donaldson invariants can be computed.

From the outset it was clear that the Donaldson polynomial invariants defined for a smooth 4-manifolds X, one for each $SU(2)$-bundle over X, were powerful tools to distinguish homeomorphic but non diffeomorphic smooth 4-manifolds. It was also clear that they were very difficult to compute. Initially, pages of detailed algebraic geometry ([**B**],[**Fr**],[**FM2**],[**MO**]), or analysis and topology ([**GM**],[**MM2**],[**SS**]) led to the computation of just the first one or two non-trivial coefficient of these invariants for special (elliptic) surfaces. This, however, was sufficient to show that the deformation type of elliptic surfaces coincides with their diffeomorphism classification ([**B**],[**MM2**], [**MO**],[**Fr**],[**FS10**]). A (recent) major breakthrough was the discovery of Peter Kronheimer and Tomas Mrowka (Theorem 1.3.1) that there are universal relations that relate these invariants defined on different moduli spaces. Then by considering the generating function determined by these invariants, i.e. the Donaldson series, these relations translated to, under a technical assumption called simple-type (defined below), a differential equation satisfied by the Donaldson series. This gave structure to the seemingly disparate Donaldson invariants. At bottom Kronheimer and Mrowka showed that for each smooth 4-manifold with

[3]Added in proof. Every minimal symplectic 4-manifold is irreducible: cf. Taubes, The Seiberg-Witten and Gromov invariants, Math. Res. Letters (2) 1996, 221-238.

classes $\kappa_1, \dots, \kappa_p \in H_2(X,\mathbf{Z})$ *and nonzero rational numbers* $a_1, \dots, a_p$ *such that*

$$\mathbf{D}_X = \exp(Q/2)\sum_{s=1}^{p} a_s e^{\kappa_s}$$

as analytic functions on $H_2(X)$. *Each of the 'basic classes'* κ_s *is characteristic, i.e.* $\kappa_s \cdot x \equiv x \cdot x \pmod 2$ *for all* $x \in H_2(X;\mathbf{Z})$.

Further, suppose $c \in H_2(X;\mathbf{Z})$. *Then*

$$\mathbf{D}_{X,c} = \exp(Q/2)\sum_{s=1}^{p}(-1)^{\frac{c^2+\kappa_s\cdot c}{2}} a_s e^{\kappa_s}$$

Here the homology class κ_s acts on an arbitrary homology class by intersection, i.e. $\kappa_s(u) = \kappa_s \cdot u$.

Theorem 1.3.2 (Kronheimer and Mrowka [**KM1, KM2**]). *If* $u \in H_2(X;\mathbf{Z})$ *is represented by an embedded surface of genus* g *with self-intersection* $u^2 \geq 0$, *then for each* s

$$2g - 2 \geq u^2 + |\kappa_s \cdot u|.$$

Theorem 1.3.3 ([**FS9**]). *Let* X *be a simply connected 4-manifold of simple type and let* $\{\kappa_s\}$ *be the set of basic classes as above. If* $u \in H_2(X;\mathbf{Z})$ *is represented by an immersed 2-sphere with* $p \geq 1$ *positive double points, then for each* s

$$2p - 2 \geq u^2 + |\kappa_s \cdot u|. \tag{1}$$

Theorem 1.3.4 ([**FS9**]). *Let* X *be a simply connected 4-manifold of simple type with basic classes* $\{\kappa_s\}$ *as above. If the nontrivial class* $u \in H_2(X;\mathbf{Z})$ *is represented by an immersed 2-sphere with no positive double points, then let*

$$\{\kappa_s \mid s = 1, \dots, 2m\}$$

be the collection of basic classes which violate the inequality (1). Then $\kappa_s \cdot u = \pm u^2$ *for each such* κ_s. *Order these classes so that* $\kappa_s \cdot u = -u^2$ (> 0) *for* $s = 1, \dots, m$. *Then*

$$\sum_{s=1}^{m} a_s e^{\kappa_s + u} - (-1)^{\frac{1+b_X^+}{2}} \sum_{s=1}^{m} a_s e^{-\kappa_s - u} = 0.$$

Problem 11. Determine whether the basic classes κ_s satisfy $\kappa_s^2 = 3\sigma(X) + 2e(X)$. [4]

Problem 12. Determine whether, up to multiples, the basic classes represent the first Chern class of some symplectic structure on X. [5]

Problem 13. Does X have big diffeomorphism group with respect to the basic classes κ_s?

Problem 14. Determine whether every simply-connected smooth 4-manifold with $b^+ > 1$ has simple-type.

Known examples suggest:

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[4] Added in proof. True for symplectic manifolds (Taubes, The Seiberg-Witten and Gromov invariant, Math. Res. Letters 2 (1995), 221-238).

[5] Added in proof. False in general, cf. footnote to Problem 5. For symplectic manifolds, there are some restrictions (cf. Taubes).

Here is another hint to show how to construct B_p in Exercise 13. The second construction begins with the configuration of $(p-1)$ 2-spheres

$$\overset{p+2}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\overset{2}{\bullet}\!\!-\!\!\!-\!\!\!-\quad\cdots\quad-\!\!\!-\!\!\!-\!\!\overset{2}{\bullet}$$

in $(p-1)\mathbf{CP}^2$ where the spheres (from left to right) represent

$$2h_1 - h_2 + \cdots + h_{p-1},\ h_1 + h_2,\ h_2 + h_3, \ldots, h_{p-2} + h_{p-1}$$

where h_i is the hyperplane class in the ith copy of $\mathbf{CP}^2$. The boundary of the regular neighborhood of the configuration is $L(p^2, p-1)$, and the classes of the configuration span $H_2(\mathbf{CP}^2; \mathbf{Q})$. The complement is the rational ball B_p.

Suppose that C_p embeds in a closed smooth 4-manifold X. Then let X_p be the smooth 4-manifold obtained by removing the interior of C_p and replacing it with B_p. We call this procedure a **rational blowdown** and say that X_p is obtained by **rationally blowing down** X. Note that $b^+(X) = b^+(X_p)$ so that rationally blowing down increases the signature while keeping b^+ fixed.

The principal gauge-theoretic result here is:

Theorem 1.3.5 ([**FS10**]). *Suppose that X has simple type and*

$$\mathbf{D}_X = \exp(Q/2)\sum_{s=1}^{n} a_s e^{\kappa_s}.$$

Let $C_p \subset X$ and let X_p be its rational blowdown. Then

$$\mathbf{D}_{X_p} = \exp(\bar{Q}/2)\sum_{s=1}^{n} b_s e^{\bar{\kappa}_s}$$

where the b_s depend only on the intersection numbers $u_i \cdot \kappa_s$, $i = 1, \ldots, p-1$, and $b_s = 0$ unless $\partial\kappa'_s \in p\mathbf{Z}_{p^2}$. The basic classes $\bar{\kappa}_s \in H_2(X_p; \mathbf{Z})$ are the unique extensions of the κ'_s. □

In short, the Donaldson invariants for X_p are completely determined by those of X; i.e there are rational blowdown formulas.

Problem 17. Determine rational blowup formulas.

Many such configurations embed in the $E(n)$ so that we can construct several manifolds in the regions graphed in Figure 1 and compute all their Donaldson invariants. It turns out that another surgical procedure called a topological logarithmic transformation of order p can be viewed as a rational blowdown of a naturally occurring $C(p)$ in $X\#_{p-1}\overline{\mathbf{CP}}^2$ so that the blowup formula and the rational blowdown formula computes the resulting Donaldson invariants (cf. [**FS10**]). In particular we can compute all the Donaldson invariants of the elliptic surfaces $E(n; m_1, \ldots m_r)$ with multiple fibers of order m_j. In particular

$$\mathbf{D}_{E(n;m_1,\ldots m_r)} = \exp\left(\frac{Q}{2}\right)\frac{(\sinh F)^{n+r-2}}{\prod_i \sinh(F/m_i)}.$$

As an amusing corollary we compute the Donaldson series of the Gompf-Mrowka examples $K = K(p_1, q_1; p_2, q_2; p_3, q_3)$ [**GM**] and find that

LECTURE 2
−2 Spheres and the Blowup Formula

2.1. Ruberman's theorem

We begin by studying the behavior of the Donaldson invariant of a 4-manifold with a homology class τ represented by an embedded 2-sphere S of self-intersection $\tau \cdot \tau = -2$. Let $\langle\tau\rangle^{\perp}$ denote $\{\alpha \in H_2(X) | \tau \cdot \alpha = 0\}$ and let

$$\mathbf{A}(\tau^{\perp}) = \mathbf{A}_X(\tau^{\perp}) = \mathrm{Sym}_*(H_0(X) \oplus \langle\tau\rangle^{\perp}).$$

Theorem 2.1.1. (Ruberman [**R**]) *Suppose that $\tau \in H_2(X;\mathbf{Z})$ with $\tau \cdot \tau = -2$ is represented by an embedded sphere S. Then for $z \in \mathbf{A}(\tau^{\perp})$, we have $D(\tau^2 z) = 2\, D_\tau(z)$.*

Proof. Write $X = X_0 \cup N$ where N is a tubular neighborhood of S, and note that ∂N is the lens space $L(2,-1) = \mathbb{R}P^3$. Since $b^+(X_0) > 0$, generically there are no reducible anti-self-dual (ASD) connections on X_0. However, since $b^+(N) = 0$ there are indeed nontrivial reducible ASD connections in complex line bundles λ^m, $m \in \mathbb{Z}$, with $c_1(\lambda^m) = m$ represented by a harmonic 2-form and with $\lambda^m|_{\partial N}$ the flat line bundle with holonomy -1 on the meridian curve, i.e. $\langle c_1(\lambda), \tau\rangle = -1$. In particular λ is determined by the non-trivial representation ζ which generates the character variety of $SU(2)$ representations of $\pi_1(\partial N) = \mathbf{Z}_2$ mod conjugacy. (Of course, ζ^{2m} is trivial, and $\zeta^{2m+1} = \zeta$.) Let $\mathbf{C} \cong \lambda^0$ be the trivial bundle. Note that for reducible connections on N, $c_1 \in H^2(N;\mathbb{Z}) \cong H_2(N;\partial N;\mathbb{Z}) \cong \mathbb{Z}$ and that $c_1(\lambda) = \gamma$ is a generator. Then $-4p_1(\lambda) = c_2(\lambda \oplus \bar{\lambda}) = -c_1^2(\lambda) = \frac{1}{2}$.

The structure of the moduli spaces $\mathcal{M}_N(\lambda^m \oplus \bar{\lambda}^m)$ and $\mathcal{M}_{X_0}$ have been studied in [**MMR**] and [**T1**]. In our setting (the boundaries are lens spaces) an anti-self-dual connection has a well-defined limiting flat connection on the restriction of the bundle to $\partial N = -\partial C_0$ and, for a generic metric on X_0, the moduli space of anti-self-dual connections $\mathcal{M}_{c_{X_0}}[\zeta^m]$ on X_0 with energy c_{X_0} and asymptotic value ζ^m contains no reducible connections and is a smooth manifold of dimension

$$\dim \mathcal{M}_{c_{X_0}}[\zeta^m] = 8c_{X_0} - \frac{3}{2}(e(X_0) + \sigma(X_0)) - \frac{h_{\zeta^m}}{2} - \frac{\rho_{\zeta^m}}{2} = 8c_{X_0} - 3(1 + b^+(X)).$$

A standard dimension counting argument (cf. [**D6**]) shows that if we choose a metric on X with long enough neck length, $\partial N \times [0,T]$, then all the intersections take place in a neighborhood $\mathcal{U}$ of the grafted moduli space $\mathcal{M}_{X_0}[\zeta]\#\{A_\lambda\}$ where A_λ is the reducible anti-self-dual connection on $\lambda \oplus \bar{\lambda}$, and $\mathcal{M}_{X_0}[\zeta]$ is the 0-dimensional cylindrical end moduli space on X_0 consisting of anti-self-dual connections which decay exponentially to the boundary value ζ. Let m_{X_0} be the signed count of points in $\mathcal{M}_{X_0}[\zeta]$. A neighborhood of A_λ in the moduli space $\mathcal{M}_N(\lambda \oplus \bar{\lambda})$ is diffeomorphic to $(\mathbf{C} \times_{S^1} SO(3))/SO(3) \cong \mathbf{C}/S^1 \cong [0,\infty)$. Here S^1 acts on $SO(3)$ so that $SO(3)/S^1 = S^2$ and on $\mathbf{C}$ with weight -2. Thus the neighborhood $\mathcal{U}$ is

$$(\tilde{\mathcal{M}}_{X_0}[\zeta] \times (\mathbf{C} \times_{S^1} SO(3)))/SO(3)$$

where "$\tilde{\mathcal{M}}_{X_0}[\zeta]$" denotes the based moduli space.

Now $\tilde{V}_1 \cap (\mathbf{C} \times_{S^1} SO(3)) = \{0\} \times_{S^1} SO(3)$, and the intersection of V_1 with all of $\mathcal{M}_X$ is

$$(\mathcal{M}_{X_0}[\zeta] \times (\{0\} \times_{S^1} SO(3)))/SO(3) = \Delta.$$

Fix a point $p \in \mathcal{M}_{X_0}[\zeta]$, let $SO(3)\cdot p$ denote its orbit in $\tilde{\mathcal{M}}_{X_0}[\zeta]$, and let

$$\Delta_p = (SO(3)\cdot p \times (\{0\} \times_{S^1} SO(3)))/SO(3) \cong S^2.$$

Identify Δ_p with a transversal in $\tilde{\Delta}_p$ and compute the intersection number $\tilde{V}_2 \cdot \Delta_p = \iota_p$. Since ι_p is independent of $p \in \mathcal{M}_{X_0}[\zeta]$, we have $D(\tau^2) = \iota_p \cdot m_{X_0}$. The constant ι_p is computed in [**FM2**] as follows. Note that $\Delta_p = \{0\} \times_{S^1} SO(3) \subset \mathbf{C} \times_{S^1} SO(3)$ is a zero-section of the $c_1 = -2$ complex line bundle over S^2 and $\tilde{V}_2$ is another section. Thus $\tilde{V}_2 \cdot \Delta_p = -2$; and so $D(\tau^2) = -2\, m_{X_0}$.

To identify the relative invariant m_{X_0}, view $\mathcal{M}_{X_0}[\zeta]$ as $\mathcal{M}_{X_0,0}[\mathrm{ad}(\zeta)]$, an $SO(3)$ moduli space. Since $\mathrm{ad}(\zeta)$ is the trivial $SO(3)$-representation, we may graft connections in $\mathcal{M}_{X_0,0}[\mathrm{ad}(\zeta)]$ to the trivial $SO(3)$ connection over N, and since $b_N^+ = 0$, there is no obstruction to doing this. We obtain an $SO(3)$ moduli space over X corresponding to an $SO(3)$ bundle over X with w_2 Poincaré dual to τ. (This is the unique nonzero class in $H^2(X;\mathbf{Z}_2)$ which restricts trivially to both N and X_0.) Thus for $z \in \mathbf{A}(\tau^\perp)$, we have $D(\tau^2 z) = \pm 2\, D_\tau(z)$. (Note that since $\tau\cdot\tau = -2$, we have $D_{-\tau} = D_\tau$.)

To determine the sign is an (important) technical matter that we refer the reader to [**FS8**].

For the case of the $SO(3)$ invariants the proof of Theorem 2.1.1 can be easily adapted to show:

Theorem 2.1.2. *Suppose that $\tau \in H_2(X;\mathbf{Z})$ with $\tau\cdot\tau = -2$ is represented by an embedded sphere S. Let $c \in H_2(X;\mathbf{Z})$ satisfy $c \cdot \tau \equiv 0 \pmod 2$. Then for $z \in \mathbf{A}(\tau^\perp)$ we have $D_c(\tau^2 z) = 2\, D_{c+\tau}(z)$.* □

These same "neck-stretching" and "dimension counting" techniques can be utilized to prove some elementary facts concerning the Donaldson invariants of blowups. These can be found, for example in [**FM2, Ko, L**]. Let X have the Donaldson invariant D, and let $\hat{X} = X\#\overline{\mathbf{CP}}^2$ have the invariant $\hat{D}$.

Exercise 14. Let $e \in H_2(\overline{\mathbf{CP}}^2;\mathbf{Z}) \subset H_2(\hat{X};\mathbf{Z})$ be the exceptional class, and let $c \in H_2(X;\mathbf{Z})$. Then for all $z \in \mathbf{A}(X)$:

1. $\hat{D}_c(e^{2k+1} z) = 0$ for all $k \geq 0$.

and $\hat{D}_{c+e}$. To see this we establish some notation. Let $\hat{X} = X\#\overline{\mathbf{CP}}^2$ and let $e \in H_2(X)$ denote the homology class of the exceptional divisor. Since $b^+_X = b^+_{\hat{X}}$, the corresponding Donaldson invariants $D = D_X$ and $\hat{D} = D_{\hat{X}}$ have their (possible) nonzero values in the same degrees (mod 4). We recursionly show that there are polynomials $B_k(x)$ satisfying

$$\hat{D}(e^k\, z) = D(B_k(x)\, z)$$

and polynomials $S_k(x)$ such that

$$\hat{D}_e(e^k) = D(S_k(x)).$$

Exercise 14 can be restated as

Exercise 15. Compute B_0, B_2, S_1, S_3, B_{2i+1}, and S_{2i}.

Now blowup X twice. The class $e_1 - e_2$ is represented by a sphere with self-intersection -2; so use Theorem 2.1.1 and Theorem 2.1.2 to show

Exercise 16. $\hat{D}_{c+e_1+e_2}((e_1+e_2)^{k+2}(e_1-e_2)^2) = 2\hat{D}_c((e_1+e_2)^{k+2})$.

Exercise 17. $\hat{D}_c((e_1+e_2)^{k+2}(e_1-e_2)^2) = 2\hat{D}_{c+e_1+e_2}((e_1+e_2)^{k+2})$.

Exercise 18. S_{k+3} is determined by $S_1,\dots,S_{k+1},B_0,\dots B_{k+2}$ for all even $k \geq 2$

Exercise 19. B_{k+4} is determined by $B_0,\dots,B_{k+2},S_1,\dots S_{k+1}$ for all even $k \geq 0$. (Hint: Expand both sides of Exercises 16 and 17)

Using Exercise 15 to start the induction, there is now a recursion scheme which determines $B_k(x)$ and $S_k(x)$. Here's a Mathematica exercise.

Exercise 20. Show $B_{12} = -512\,x^4 - 960\,x^2 - 408$ and

$$\begin{aligned} B_{30}(x) = {} & 134{,}217{,}728\,x^{13} + 4{,}630{,}511{,}616\,x^{11} + 68{,}167{,}925{,}760\,x^9 \\ & - 34{,}608{,}135{,}536{,}640\,x^7 - 39{,}641{,}047{,}695{,}360\,x^5 \\ & - 9{,}886{,}101{,}110{,}784\,x^3 + 543{,}185{,}367{,}552\,x \end{aligned}$$

To actually find a closed form for $B_k(x)$ and $S_k(x)$ requires a bit more work and for this we use a result for embedded 2-spheres of self-intersection -3. Again, it's proof similar to that of Theorem 2.1.1.

Theorem 2.2.1. *Suppose that $\tau \in H_2(X;\mathbf{Z})$ is represented by an embedded 2-sphere S with self-intersection -3. Let $\omega \in H_2(X;\mathbf{Z})$ satisfy $\omega \cdot \tau \equiv 0 \pmod 2$. Then for all $z \in \mathbf{A}(\tau^\perp)$ we have*

$$D_\omega(\tau\, z) = -D_{\omega+\tau}(\,z).$$

To conclude we use the relations given in Theorem 2.1.1, Theorem 2.1.2 and Theorem 2.2.1 to obtain a relation key to solving the recursion scheme given in above. This relation was first proved by Wojciech Wieczorek using different methods. His proof will appear in his thesis [**Wk**].

and

$$S(x,t) = \sum_{k=0}^{\infty} S_k(x) \frac{t^k}{k!}.$$

This will amount to solving an ordinary differential equation.

2.4. The ODE

Note:

$$\begin{aligned} \frac{d^n}{dt^n} \hat{D}(\exp(te)\, z) &= \hat{D}(\, z \sum e^{k+n} \frac{t^k}{k!}) = D(B_{k+n}(x) \frac{t^k}{k!}\, z) \\ &= \frac{d^n}{dt^n} D(B(x,t)\, z) = D(B^{(n)}(x,t)\, z) \end{aligned} \tag{2}$$

where the last differentiation is with respect to t. On $\bar{X} = X \# 2\overline{\mathbf{CP}}^2$, we get $\bar{D}(\exp(t_1 e_1 + t_2 e_2) z) = D(B(x,t_1)\, B(x,t_2) z)$. Now apply Corollary 2.2.2 to $e_1 - e_2 \in H_2(\bar{X};\mathbf{Z})$. Since for any $t \in \mathbf{R}$ the class $te_1 + te_2 \in \langle e_1 - e_2 \rangle^{\perp}$, we have the equation

$$\begin{aligned} &\bar{D}(\exp(te_1 + te_2)\, (e_1 - e_2)^4\, z) + 4\, \bar{D}(x \, \exp(te_1 + te_2)\, (e_1 - e_2)^2\, z) \\ &\qquad\qquad + 4\, \bar{D}(\exp(te_1 + te_2)\, z) = 0 \end{aligned} \tag{3}$$

But, for example,

$$e_1^4 \, \exp(te_1 + te_2) = \left(\sum e_1^{k+4} \frac{t^k}{k!}\right) \left(\sum e_2^k \frac{t^k}{k!}\right) = \frac{d^4}{dt^4}(\exp(te_1))\, \exp(te_2)$$

Arguing similarly and using (2)

$$\begin{aligned} &\bar{D}(\exp(te_1 + te_2)\, (e_1 - e_2)^4\, z) \\ &= D((2\, B^{(4)}(x,t)\, B(x,t) - 8\, B'''(x,t)\, B'(x,t) + 6\, (B''(x,t))^2)\, z) \\ &= 2\, D((B^{(4)}\, B - 4\, B'''\, B' + 3\, (B'')^2)\, z) \end{aligned}$$

where $B = B(x,t)$. Completing the expansion of (3) we get

$$2\, D((B^{(4)}\, B - 4\, B'''\, B' + 3\, (B'')^2 + 4x\, (B'' B - (B')^2) + 2\, B^2)\, z) = 0$$

for all $z \in \mathbf{A}(X)$. This means that the expression

$$B^{(4)}\, B - 4\, B'''\, B' + 3\, (B'')^2 + 4x\, (B'' B - (B')^2) + 2\, B^2$$

lies in the kernel of $D : \mathbf{A}(X) \to \mathbf{R}$.

Thus the "blowup function" $B(x,t)$ satisfies the differential equation

$$B^{(4)}\, B - 4\, B'''\, B' + 3\, (B'')^2 + 4x\, (B'' B - (B')^2) + 2\, B^2 = 0$$

modulo the kernel of D. Of course, the fact that this equation holds only modulo the kernel of D is really no constraint, since our interest in $B(x,t)$ comes from the equation $\hat{D}(\exp(te) z) = D(B(x,t) z)$.

Exercise 21. Modulo the kernel of D, the logarithm $f(t)$ of $B(x,t)$ satisfies the differential equation

$$f^{(4)} + 6\, (f'')^2 + 4x f'' + 2 = 0 \tag{4}$$

with the initial conditions $f(0) = f'(0) = f''(0) = f'''(0) = 0$. (Hint: Let $B = \exp(f(t))$. The initial conditions follow from Exercise 15.)

(which follow easily from [**Ak**, p.199]). Using them, our formula for $B(x,t)$ becomes

$$B(x,t) = e^{-\frac{t^2x}{6}} e^{-\eta_3 t} \frac{\sigma(t+\omega_3)}{\sigma(\omega_3)}.$$

The above addition formula for the sigma-function implies that

$$\sigma(t+\omega_3) = \sigma((t-\omega_3)+2\omega_3) = -e^{2\eta_3 t}\sigma(t-\omega_3).$$

Thus

$$B(x,t) = -e^{-\frac{t^2x}{6}} e^{\eta_3 t} \frac{\sigma(t-\omega_3)}{\sigma(\omega_3)} = e^{-\frac{t^2x}{6}}\sigma_3(t),$$

the last equality by the definition of the quasi-periodic function σ_3. In conclusion,

Theorem 2.5.1. ([**FS8**]) *Modulo the kernel of D, the blowup function $B(x,t)$ is given by the formula*

$$B(x,t) = e^{-\frac{t^2x}{6}}\sigma_3(t). \qquad \square$$

The indexing of the Weierstrass functions σ_i depends on the ordering of the roots e_i of the equation $4s^3 - g_2 s - g_3 = 0$; the sigma-function we are using corresponds to the root $-\frac{x}{3}$.

These ideas also show

Theorem 2.5.2. ([**FS8**]) *Modulo the kernel of D, the blowup function $S(x,t)$ is given by the formula*

$$S(x,t) = e^{-\frac{t^2x}{6}}\sigma(t).$$

2.6. The simple-type condition

An elliptic function is a doubly-periodic function, so it is interesting to see what happens when one of the periods goes to infinity. This gives a degenerate case of the associated Weierstrass functions. The squares k^2, k'^2 of the modulus and complementary modulus of our Weierstrass functions are given by

$$k^2 = \frac{x-\sqrt{x^2-4}}{x+\sqrt{x^2-4}} \qquad k'^2 = \frac{2\sqrt{x^2-4}}{x+\sqrt{x^2-4}}.$$

(All the formulas involving elliptic functions can be found in [**Ak**].) These degenerate when $x=2$ and then $k^2=1$ and $k'^2=0$. The corresponding complete elliptic integrals of the first kind are

$$\begin{aligned} K &= \int_0^1 \frac{ds}{\sqrt{(1-s^2)(1-k^2s^2)}} = \int_0^1 \frac{ds}{1-s^2} \\ K' &= \int_0^1 \frac{ds}{\sqrt{(1-s^2)(1-k'^2s^2)}} = \int_0^1 \frac{ds}{\sqrt{1-s^2}} \end{aligned}$$

Theorem 2.6.2. *([FS8]) If X has c-simple type,*

$$\hat{\mathbf{D}}_c = \mathbf{D}_c \cdot e^{-\frac{E^2}{2}} \cosh E\,.$$
$$\hat{\mathbf{D}}_{c+e} = -\mathbf{D}_c \cdot e^{-\frac{E^2}{2}} \sinh E\,. \qquad \square$$

Exercise 25. If X has simple type, determine the basic classes of $X\#\overline{\mathbf{CP}}^2$.

All of this from relations involving spheres with self-intersection -2. There is another application that we will discuss in the next lecture.

LECTURE 3
Simple-Type Criteria and Elliptic Surfaces

3.1. Manifolds with big diffeomorphism group

Using Theorem 2.1.1 we show that for many manifolds the homogeneous parts of D_X are related, show these manifolds have simple-type. For a 4-manifold X and a class $\kappa \in H_2(X;\mathbf{Z})$, let $\mathrm{Diff}_\kappa(X)$ be the group of orientation-preserving diffeomorphisms f of X which satisfy $f_*(\kappa) = \kappa$. Also, let $\mathrm{Aut}(X)$, be the group of automorphisms of $H_2(X;\mathbf{Z})$ which preserve the intersection form Q. Then X is said to have a *big diffeomorphism group* with respect to κ if the image of $\mathrm{Diff}_\kappa(X)$ in $\mathrm{Aut}(X)$ has finite index. For example, the simply connected minimal elliptic surfaces with $p_g \geq 1$ have a big diffeomorphism group with respect to their canonical class **[FM2]**. It follows from the assumption of big diffeomorphism group with respect to κ that for each d, the degree d homogeneous part $D^{(d)}_{X,c}$ of the Donaldson invariants $D_{X,c}$ and $D_{X,c}(\frac{x}{2})$ (and hence $\hat{D}_{X,c}$) are polynomials in the intersection form Q and the class κ when viewed as linear maps $\mathrm{Sym}_*(H_2(X)) \to \mathbf{R}$. If $\frac{1}{2}(1 + b^+_X) \equiv 0 \pmod 2$ we can then write

$$\begin{aligned}\frac{1}{(2d)!}\hat{D}^{(2d)}_X &= c_0\frac{Q^d}{2^d d!} + c_2\frac{Q^{d-1}}{2^{d-1}(d-1)!}\frac{\kappa^2}{2!} + \dots \\ &+ c_{2t}\frac{Q^{d-t}}{2^{d-t}(d-t)!}\frac{\kappa^{2t}}{(2t)!} + \dots + c_{2d}\frac{\kappa^{2d}}{(2d)!}\end{aligned}$$

and if $\frac{1}{2}(1 + b^+_X) \equiv 1 \pmod 2$ we can write

$$\begin{aligned}\frac{1}{(2d+1)!}\hat{D}^{(2d+1)}_X &= c_1\frac{Q^d}{2^d d!}\kappa + \dots \\ &+ c_{2t+1}\frac{Q^{d-t}}{2^{d-t}(d-t)!}\frac{\kappa^{2t+1}}{(2t+1)!} + \dots + c_{2d+1}\frac{\kappa^{2d+1}}{(2d+1)!}.\end{aligned}$$

Our next Proposition states that if X contains an embedded 2-sphere of self-intersection -2 which is orthogonal to κ, the coefficients c_j are independent of the homogeneous degree. Related results were first observed by Peter Kronheimer (unpublished).

Thus $c_0' = -c_0''$ and so $c_0 = c_0''$. In a similar fashion we get that $c_{2j} = c_{2j}''$ for all j.

Finally consider $\hat{D}_X^{(2d-2)} = D_X^{(2d)}(\frac{x}{2})$. Because X has a big diffeomorphism group we can write

$$\frac{1}{(2d-2)!}\hat{D}_X^{(2d-2)} = \hat{c}_0 \frac{Q^{d-1}}{2^{d-1}(d-1)!} + \hat{c}_2 \frac{Q^{d-2}}{2^{d-2}(d-2)!}\frac{\kappa^2}{2!} + \dots$$

Proposition 2.2.2 implies that

$$D_X(\alpha^{2d}\sigma^4) = -8\, D_X(\alpha^{2d}\sigma^2 \frac{x}{2}) - 4\, D_X(\alpha^{2d}).$$

Expanding as above, and using the fact that $c_0'' = c_0$, we get $\hat{c}_0 = c_0$, and continuing as above, $\hat{c}_i = c_i$ for all i. This completes the proof in the $SU(2)$ case with $\frac{1}{2}(1+b_X^+) \equiv 0 \pmod 4$. A similar proof suffices when $\frac{1}{2}(1+b_X^+) \not\equiv 0 \pmod 4$, and the same proof also works in the $SO(3)$ case.

3.2. A simple-type criteria

A related argument shows that these manifolds have simple-type.

Theorem 3.2.1. *([FS9]) Let X be a simply connected 4-manifold which has a big diffeomorphism group with respect to a class $\kappa \in H_2(X;\mathbf{Z})$, and let $\omega \in H_2(X;\mathbf{Z})$. Suppose that $\sigma \in H_2(X;\mathbf{Z})$ is represented by an embedded 2-sphere of square -2 such that $\sigma \cdot \kappa = 0$, and $\sigma \cdot \omega \equiv 0 \pmod 2$. Then X has ω-simple type. If $\omega^2 + \frac{1}{2}(1+b^+) \equiv 0 \pmod 2$ then*

$$\mathbf{D}_{X,\omega} = \exp(Q/2)\sum c_{2i,\omega}\frac{\kappa^{2i}}{(2i)!},$$

and if $\omega^2 + \frac{1}{2}(1+b^+) \equiv 1 \pmod 2$ then

$$\mathbf{D}_{X,\omega} = \exp(Q/2)\sum c_{2i+1,\omega}\frac{\kappa^{2i+1}}{(2i+1)!}.$$

Proof. For simplicity of notation, consider the $SU(2)$ case $\omega = 0$ with $\frac{1}{2}(1+b_X^+) \equiv 0 \pmod 4$ as above. Then Proposition 3.1.1 shows that for each $d \equiv \frac{1}{2}(1+b^+) \pmod 4$ the homogeneous invariants $D_X^{(2d+4)}(\frac{x}{2}) = \hat{D}_X^{(2d+2)}$ and $D_X^{(2d)}$ share the same coefficients in that

$$\begin{aligned}\frac{1}{2d!}D_X^{(2d)} &= \sum_{i=0}^{d} c_{2i}\frac{Q^{d-i}}{2^{d-i}(d-i)!}\frac{\kappa^{2i}}{(2i)!}\\ \frac{1}{(2d+2)!}D_X^{(2d+4)}(\frac{x}{2}) &= \sum_{i=0}^{d+1} c_{2i}\frac{Q^{d+1-i}}{2^{d+1-i}(d+1-i)!}\frac{\kappa^{2i}}{(2i)!}\end{aligned}$$

The technique of Proposition 3.1.1 also shows that

$$\frac{1}{(2d+4)!}D_X^{(2d+8)}(\frac{x^2}{4}) = \sum_{i=0}^{d+2} c_{2i}\frac{Q^{d+2-i}}{2^{d+2-i}(d+2-i)!}\frac{\kappa^{2i}}{(2i)!},$$

independent of degree. Thus also,

$$\frac{1}{2d!}D_X^{(2d+4)}(\frac{x^2}{4}) = \sum_{i=0}^{d} c_{2i}\frac{Q^{d-i}}{2^{d-i}(d-i)!}\frac{\kappa^{2i}}{(2i)!} = \frac{1}{(2d)!}D_X^{(2d)};$$

$E(n-1)\#\overline{\mathbf{CP}}^2$ and using the blowup formula. Here's where a little topology helps out. It turns out that $E(n)$, $n \geq 2$, can be decomposed as

$$E(n) = B(2,3,11) \cup W(n).$$

That is,

Exercise 26. The Milnor fiber $B(2,3,11)$ embeds in $E(n)$, $n \geq 2$. (Hint: The Milnor fibers are nested: $B(p_1, p_2, \dots, p_r) \subset B(p_1', p_2', \dots, p_r')$ whenever $p_j \leq p_j'$ for all j and $E(n) = B(2,3,6n-1) \cup G_n$, where G_n is a so-called Gompf nucleus [**G1**].)

Now $\partial B(2,3,11)$ bounds another interesting manifold, $C(2,3,11)$ obtained as the union $C(2,3,11) = B(2,3,5) \cup D$, where D is constructed by attaching one 2-handle to $\Sigma(2,3,5)$ This follows since $-\Sigma(2,3,5)$ is -1 surgery on the left hand trefoil knot and $-\Sigma(2,3,11)$ is $-\frac{1}{2}$ surgery on the same knot.

Exercise 27. $C(2,3,11) \cup W(n) = E(n-1)\#\overline{\mathbf{CP}}^2$.

This is our inductive construction: replace $C(2,3,11)$ in $E(n-1)\#\overline{\mathbf{CP}}^2$ by $B(2,3,11)$ to obtain $E(n)$.

We are after the following computation:

Theorem 3.5.1. *([**FS9**]) The Donaldson series of the elliptic surfaces $E(n)$ are given by*

$$\mathbf{D}_{E(n),c} = (-1)^{\frac{1}{2}(c^2+(n-2)c\cdot f)} \exp(Q/2) \sinh^{n-2}(f) \qquad \text{if} \quad c\cdot f \equiv 0 \pmod 2,$$
$$\mathbf{D}_{E(n),c} = (-1)^{\frac{1}{2}(c^2+(n-2)c\cdot f)} \exp(Q/2) \cosh^{n-2}(f) \qquad \text{if} \quad c\cdot f \equiv 1 \pmod 2.$$

where $f \in H_2(E(n);\mathbf{Z})$ is the homology class of the fiber.

This result has also been obtained by Paulo Lisca [**Li**] and Kronheimer-Mrowka [**KM1**].

Exercise 28. Determine the basic classes for $E(n)$.

The idea of the proof is simple; suppose $c \cdot f \equiv 0 \pmod 2$. Since elliptic surfaces have a big diffeomorphism group with respect to their fiber and simple-type, it follows by Theorem 3.1.1 that $\mathbf{D}_{E(n)} = \exp(Q/2)\mathbf{K}$ where $\mathbf{K}$ is a formal power series in f. Assume inductively that

$$\mathbf{D}_{E(n),c} = (-1)^{\frac{1}{2}(c^2+(n-2)c\cdot f)} \exp(Q/2) \sinh^{n-2}(f)$$

$K3 = E(2)$ starts the induction. The blow-up formula says that

$$\mathbf{D}_{E(n)\#\overline{\mathbf{CP}}^2,c} = (-1)^{\frac{1}{2}(c^2+(n-2)c\cdot f)} \exp(Q/2) \sinh^{n-2}(f) \cosh(e)$$

But $C(2,3,11) \cup W(n) = E(n-1)\#\overline{\mathbf{CP}}^2$. The thing to show is that by replacing $C(2,3,11)$ by $B(2,3,11)$ replaces $\cosh(e)$ by $\sinh(f)$. Since the Floer homology, $HF_*(\Sigma(2,3,11))$ is a copy of $\mathbf{Z}$ in odd dimensions and vanishes in even dimensions [**FS3**], this becomes a fun exercise. A technical point here is that when you stretch the neck of $E(n-1)\#\overline{\mathbf{CP}}^2$ along $\Sigma(2,3,11)$, the trivial connection doesn't occur as asymptotic value –guaranteeing the fact that reducible connections in the negative definite manifold $C(2,3,11)$ do not complicate the picture. The proof here is interesting, and is again typical of many of our arguments in [**FS9**].

LECTURE 4
Elementary Rational Blowdowns

4.1. Elementary rational blowdowns

There are two directions in which to generalize our basic relations Theorem 2.1.1 and Theorem 2.2.1. The first, as we have mentioned before, is to prove relations for spheres of arbitrary self-intersection. This is done in [**FS9**] and results in Theorem 2.3.1. This in turn implies the general structure theorems and adjunction formulas for the Donaldson invariants mentioned in the first lecture. Here's another fruitful generalization. Suppose S is a sphere with self-intersection -4 and let N (=C_2)) be a tubular neighborhood of S so that $X = X_0 \cup N$.

Exercise 29. ∂N bounds the Euler class -2 non-orientable S^1-bundle over $\mathbf{RP}^2$, a rational ball B_2.

Let X_2 be the result of replacing N by B_2.

Proposition 4.1.1. *For all $z \in \mathbf{A}(X_2)$,*

$$D_{X_2}(z) = D_X(z) - D_{X,[S]}(z).$$

Proof. We outline the proof. Let $L = \partial N$ and $\chi(L) = \{\zeta^0 = \vartheta, \zeta, \zeta^2, \zeta^3\}$ be the character variety of L. The only characters which extend across B_2 are ϑ and ζ^2. Thus $D_{X_2}(z) = \pm D_{X_0}[\vartheta](z) \pm D_{X_0}[\zeta](z)$. By stretching the neck of X along L, dimension counting shows that $D_{X_0}[\vartheta](z) = D_X(z)$ (for the dimension of $\mathcal{M}_N = -3$). To determine $D_{X_0}[\zeta](z)$, we do as in Theorem 2.1.1 and glue in the trivial bundle over N with a twist to obtain $D_{X_0}[\zeta](z) = \pm D_{X,[S]}(z)$. Again, a delicate (but important) point is to get the correct signs. For this we refer to [**FS10**]. Now the fun begins!

Consider $E(4)$ with S being the section. Since

$$\mathbf{D}_{E(4)} = \exp(\frac{Q}{2}) \sinh^2(F)$$

and

$$\mathbf{D}_{E(4),S} = (-1)^{\frac{S^2+2F\cdot S}{2}} \exp(\frac{Q}{2}) \cosh^2(F) = -\exp(\frac{Q}{2}) \cosh^2(F),$$

we have $f = p f_p$. Continue this process on other fibers, but to insure that the resulting manifold is simply-connected we can take at most two log-transforms with multiplicities that are pairwise relatively prime. Let the orders be p and q and denote the result by $E(n;p,q)$.

The homology class f of the fiber of $E(n)$ can be represented by an immersed sphere with one positive double point (a nodal fiber). Blow-up this double point (i.e. take the proper transform of f) so that the class $f - 2e_1$ (where e_1 is the homology class of the exceptional divisor) is represented by an embedded sphere with square -4. This is just the configuration C_2. Now the exceptional divisor intersects this sphere in two positive points. Blow-up one of these points, i.e. again take a proper transform. There results the homology classes $u_0 = f - 2e_1 - e_2$ and $u_1 = e_1 - e_2$ which is just the configuration C_3. Continuing in this fashion, C_p naturally embeds in $N\#_{p-1}\overline{\mathbf{CP}}^2 \subset E(n)\#_{p-1}\overline{\mathbf{CP}}^2$.

An important example of a rational blowdown is:

Theorem 4.2.1 ([**FS10**]). *The rational blowdown of the above configuration $C_p \subset E(n)\#(p-1)\overline{\mathbf{CP}}^2$ is diffeomorphic $E(n;p)$.*

When $p = 2$, this was first observed by Gompf [**G3**]

We can now easily compute $\mathbf{D}_{E(2;2)}$ on $X \setminus C_2$.

$$\begin{aligned}
\mathbf{D}_{E(2;2)} &= \mathbf{D}_{E(2)\#\overline{\mathbb{C}P}^2} - \mathbf{D}_{E(2)\#\overline{\mathbb{C}P}^2, f-2e} \\
&= \exp(\frac{Q}{2})\cosh(E) - (-1)^{\frac{(f-2e)^2 + E\cdot(f-2e)}{2}}\exp(\frac{Q}{2})\cosh(E) \\
&= 2\exp(\frac{Q}{2})\cosh(E).
\end{aligned}$$

While E is not a cohomology class in $E(2;2)$, the appropriate class is $E + \frac{1}{2}(F - 2E) = \frac{F}{2}$ (since we can add multiples of classes in the neighborhood of the -4 sphere), i.e. the multiple fiber. So

$$\mathbf{D}_{E(2;2)} = 2\exp(\frac{Q}{2})\cosh(\frac{F}{2}) = \exp(\frac{Q}{2})\frac{\sinh(F)}{\sinh(\frac{F}{2})} \tag{9}$$

4.3. The basic computational theorem

To compute $\mathbf{D}_{X_p}$ for large p requires an important preliminary observation about the Donaldson series. For $u \in H_2(X)$ and $F \in \mathbf{A}^*(X)$, the interior product

$$\iota_u F(v) = (\deg(v) + 1)F(uv)$$

is a derivation. On the formal power series $\mathbf{F}$ on $H_2(X)$ defined by $\mathbf{F}(\alpha) = F(\exp(\alpha))$ this induces

$$\partial_u \mathbf{F}(\alpha) = F(u\exp(\alpha)).$$

This is just the formal derivative of $\mathbf{F}$ in the direction u. Similarly, for higher order derivatives, $\partial_u^k \mathbf{F}(\alpha) = F(u^k \exp(\alpha))$. Note that the linearity of F implies that $\partial_{u+v}\mathbf{F} = \partial_u \mathbf{F} + \partial_v \mathbf{F}$.

An induction argument shows that

$$\partial_u^{2k}\exp(Q/2) = \exp(Q/2)\sum_{t=0}^{k}(u\cdot u)^{k-t}\tilde{u}^{2t}\binom{2k}{2t}\frac{(2k-2t)!}{2^{k-t}(k-t)!} \tag{10}$$

We begin by reexpressing these equations. Let $\{\omega_i\}$ be a standard basis for $\mathbf{Q}^{p-1}$, and let A be the $(p-1)\times(p-1)$ matrix whose ith row vector is

$$\begin{aligned} A_i &= \omega_{p-(i+1)} - \omega_{p-i}, \quad i = 1,\dots,p-2 \\ A_{p-1} &= -2\omega_1 - \omega_2 - \cdots - \omega_{p-1} \end{aligned}$$

We have $u_i = A^t(\omega_i)\cdot \mathbf{e}$ and $u_{p-1} = f + A^t(\omega_{p-1})\cdot\mathbf{e}$, where $\mathbf{e} = (e_1,\dots,e_{p-1})$. Our linear system is equivalent to

$$P\mathbf{x} = A\boldsymbol{\epsilon}_J$$

where $\mathbf{x} = (x_1,\dots,x_{p-1})$ and $\boldsymbol{\epsilon}_J = (\epsilon_{J,1},\dots,\epsilon_{J,p-1})$. (The matrix P is the plumbing matrix for C_p.) Hence $\mathbf{x} = P^{-1}A\boldsymbol{\epsilon}_J$.

We claim that $P(A^t)^{-1} = -A$. This can be checked on the basis

$$\{\omega_2-\omega_1,\dots,\omega_{p-1}-\omega_{p-2},\omega_{p-1}\}$$

using

$$A(\omega_i) = -\omega_{p-1} - \omega_{p-(i+1)} + \omega_{p-i},\ 2\le i\le p-1\ (\omega_0 = 0),$$

$$A(\omega_1) = -2\omega_{p-1} + \omega_{p-2}$$

$$P(\omega_i) = \omega_{i+1} - 2\omega_i + \omega_{i-1},\ i\neq p-1$$

$$P(\omega_{p-1}) = -(p+2)\omega_{p-1} + \omega_{p-2}.$$

It follows that $A^tP^{-1}A = -I$. Thus

$$\kappa_J + \zeta = \kappa_J + \sum x_iu_i = (\epsilon_J + A^t\mathbf{x})\cdot\mathbf{e} + x_{p-1}f = (\boldsymbol{\epsilon}_J - \boldsymbol{\epsilon}_J)\cdot\mathbf{e} + x_{p-1}f = x_{p-1}f.$$

To compute x_{p-1} note that

$$A\boldsymbol{\epsilon}_J = (\epsilon_{J,p-2} - \epsilon_{J,p-1}, \epsilon_{J,p-3} - \epsilon_{J,p-2},\dots,\epsilon_{J,1} - \epsilon_{J,2}, -2\epsilon_{J,1} - \epsilon_{J,2} - \cdots - \epsilon_{J,p-1})$$

so that if $(P^{-1})_{p-1}$ denotes the bottom row of P^{-1}:

$$x_{p-1} = (P^{-1})_{p-1}(A\boldsymbol{\epsilon}_J) = -\frac{1}{p^2}(1,2,\dots,p-1)\cdot(A\epsilon_J) = \frac{1}{p}\sum\epsilon_{J,i} = \frac{1}{p}|J|.$$

Thus $\kappa_J|_{X_c} = \kappa_J + \zeta = \frac{1}{p}|J|f$ as forms: $H_2(X_c;\mathbf{Z})\to\mathbf{Z}$. The homology class $\kappa_J+\zeta$ is in fact an integral class $\bar\kappa_J = |J|f_p \in H_2(X_p;\mathbf{Z})$ which is the unique extension of $\kappa_J|_{X_c}$

In an arbitrary smooth 4-manifold X, define a *nodal fiber* to be an immersed 2-sphere S with one singularity, a positive double point, such that the regular neighborhood of S is diffeomorphic to the regular neighborhood of a nodal fiber in an elliptic surface. (There need not be any associated ambient fibration of X.) Given such a nodal fiber S, one can perform a 'log transform' of multiplicity p by blowing up to get $C_p \subset X\#(p-1)\overline{\mathbf{CP}}^2$ with $u_{p-1} = S - 2e_1 - e_2 - \cdots - e_{p-1}$, and then blowing down C_p. We denote the result of this process by X_p.

Throughout, we use the following notation. If X has simple type, and

$$\mathbf{D}_X = \exp(Q/2)\sum a_s e^{\kappa_s},$$

then we write $\mathbf{K}_X = \sum a_s e^{\kappa_s}$.

This means that the sum of the coefficients of the expression for $\mathbf{D}_{X_p}$ in Proposition 4.4.2 is $\sum_J b_J = p$.

An interesting observation is that if p is any positive odd integer, then a multiplicity $2p$ log transform can be obtained as the result of either a multiplicity p log transform on a nodal fiber of multiplicity 2, or by a multiplicity 2 log transform on a nodal fiber of multiplicity p. Thus

$$\begin{aligned}\mathbf{D}_{E(n;2p)} &= \exp(Q/2)(e^{f_2}+e^{-f_2})(b_{p,0}+\sum_{i=1}^{(p-1)/2} b_{p,2i}(e^{2if_2/p}+e^{-2if_2/p}))\\ &= \exp(Q/2)(b_{p,0}+\sum_{i=1}^{(p-1)/2} b_{p,2i}(e^{2if_p}+e^{-2if_p}))(e^{f_p/2}+e^{-f_p/2})\end{aligned}$$

since we already know the formula for a log transform of multiplicity 2. We compare coefficients using $f_2 = pf_{2p}$, $f_p/2 = f_{2p}$, and $f_p = 2f_{2p}$.

Assume for the sake of definiteness that $p \equiv 1 \pmod 4$ and let $r = (p-1)/4$. In the top expansion, the coefficient of $e^{\pm pf_{2p}}$ is $b_{p,0}$ and $b_{p,2j}$ is the coefficient of $e^{\pm(p+2j)f_{2p}}$ and $e^{\pm(p-2j)f_{2p}}$. In the second expansion, the coefficient of $e^{\pm f_{2p}}$ is $b_{p,0}$, and $b_{p,2j}$ is the coefficient of $e^{\pm(4j-1)f_{2p}}$ and $e^{\pm(4j+1)f_{2p}}$. To simplify notation, let $(m)_1$ be the coefficient of $e^{mf_{2p}}$ in the top expansion and $(m)_2$ its coefficient in the bottom expansion. Then,

$$\begin{aligned}b_{p,0} &= (p)_1 = (p)_2 = b_{p,2r} = (p-2)_2 = (p-2)_1 = b_{p,2} = (p+2)_1\\ &= (p+2)_2 = b_{p,2(r+1)} = (p+4)_2 = (p+4)_1 = b_{p,4} = (p-4)_1 = (p-4)_2\\ &\qquad = b_{p,2(r-1)} = (p-6)_2 = (p-6)_1 = b_{p,6} = \cdots\end{aligned}$$

and we see inductively that when p is odd, all the $b_{p,2i}$ are equal. Lemma 4.4.4 now implies

$$b_{p,0} + 2\sum_{i=1}^{(p-1)/2} b_{p,2i} = p.$$

It follows that each $b_{p,2i} = 1$, $i = 0, \dots, (p-1)/2$.

Similarly, if p is even, let $q = p-1$. Expanding $\mathbf{D}_{E(n;pq)}$ we see that all $b_{p,2i-1}$, $i = 1, \dots, p/2$ are equal; and so again each $b_{p,2i-1} = 1$.

Theorem 4.4.5. *Let X be a 4-manifold of simple type and suppose that X contains a nodal fiber S orthogonal to all its basic classes. Then*

$$\mathbf{D}_{X_p} = \exp(Q_{X_p}/2)K_X \cdot \frac{\sinh(S)}{\sinh(S/p)}.$$

Proof. If, e.g., p is odd,

$$\begin{aligned}\mathbf{D}_{X_p} &= \exp(Q_{X_p}/2)\mathbf{K}_X \cdot (1 + 2\cosh(2S/p) + 2\cosh(4S/p) + \cdots\\ &+ 2\cosh((p-1)S/p))\\ &= \exp(Q_{X_p}/2)\mathbf{K}_X \cdot \frac{\sinh(S)}{\sinh(S/p)}\end{aligned}$$

As a result we have the calculation of the Donaldson series for all simply connected elliptic surfaces with $p_g \geq 1$.

LECTURE 5
Taut Configurations and Horikowa Surfaces

5.1. Taut configurations

Consider a 4-manifold X of simple type containing the configuration C_p. By Theorem 1.3.3 for each 2-sphere u_i in C_p and each basic class κ of X, we have

$$-2 \geq u_i^2 + |u_i \cdot \kappa| \tag{15}$$

except in the special case described in Theorem 1.3.4 where $0 \geq u_i^2 + |u_i \cdot \kappa|$. The only examples known where the special case arises are in blowups. This was the situation in the previous section where we studied log transforms. In this section, we assume that we are not in the special case. We say that a configuration is *tautly embedded* if (15) is satisfied for each u_i of the configuration and each basic class κ of X. Thus, if C_p is tautly embedded, then for every basic class κ, $u_i \cdot \kappa = 0$ for $i = 1, \ldots, p-2$ and $|u_{p-1} \cdot \kappa| \leq p$.

Theorem 5.1.1. *Suppose that X is of simple type and contains the tautly embedded configuration C_p. If*

$$\mathbf{D}_X = \exp(Q/2) \sum a_s e^{\kappa_s}$$

then the rational blowdown X_p satisfies

$$\mathbf{D}_{X_p} = \exp(\bar{Q}/2) \sum \bar{a}_s e^{\bar{\kappa}_s}$$

where

$$\bar{a}_s = \begin{cases} 2^{p-1} a_s, & |u_{p-1} \cdot \kappa_s| = p \\ 0, & |u_{p-1} \cdot \kappa_s| < p \end{cases}$$

Furthermore, if $|u_{p-1} \cdot \kappa_s| = p$ then $\bar{\kappa}_s^2 = {\kappa_s}^2 + (p-1)$.

Proof. An algebraic topological observation is that if $\kappa_s \cdot u_{p-1} \neq 0, \pm p$ then $\bar{a}_s = 0$. For $\kappa_s \cdot u_{p-1} = 0$ note that since the κ_s are characteristic p must be even. For the model, consider Example 4; the smooth 4-manifold $Y = E(p+1) \# (p-1)\overline{\mathbf{CP}}^2$ contains the configuration C'_p with $u'_i = e_{p-i} - e_{p-(i+1)}$ for $1 \leq i \leq p-2$ and $u'_{p-1} = s + e_1$ where s a section of $E(p+1)$ (oriented so that $s \cdot f = 1$) which

$n-2$, $u_j \cdot f = 0$. Furthermore, the rational blowdown of this pair of configurations is the Horikawa surface $H(n)$.

Proof. It follows from our description of $H(n)$ that there is a decomposition

$$H(n) = B_{n-2} \cup \tilde{D}_{n-2} \cup B_{n-2}$$

where $\tilde{D}_{n-2}$ is the branched cover of D_{n-2}. Rationally blow up each B_{n-2}; this is then the 2–fold branched cover of $\mathbf{F}_{n-3}$ with B_{n-2} blown up. The result is the complex surface $C_{n-2} \cup \tilde{D}_{n-2} \cup C_{n-2}$ which, by computing characteristic numbers, is just $E(n)$. The first case $n = 4$ gives the example $H(4) = W_2$ above. The Horikawa surfaces $H(n)$ lie on the Noether line $5c_1^2 - c_2 + 36 = 0$, and of course the elliptic surfaces $E(n)$ lie on the line $c_1^2 = 0$ in the plane of coordinates (c_1^2, c_2). Let $Y(n)$ be the simply connected 4-manifold obtained from $E(n)$ by blowing down just one of the configurations C_{n-2}. Then $c_1(Y(n))^2 = n-3$ and $c_2(Y(n)) = 11n + 3$; so $Y(n)$ lies on the bisecting line $11c_1^2 - c_2 + 36 = 0$. The calculation of Donaldson invariants of $Y(n)$ and $H(n)$ follows directly from Theorem 5.1.1.

Proposition 5.2.2. *The Donaldson invariants of $Y(n)$ and $H(n)$ are:*

$$\mathbf{D}_{Y(n)} = \begin{cases} \exp(Q/2)\sinh(\lambda_n), & n \text{ odd} \\ \exp(Q/2)\cosh(\lambda_n), & n \text{ even} \end{cases}$$

$$\mathbf{D}_{H(n)} = \begin{cases} 2^{n-3}\exp(Q/2)\sinh(\kappa_n), & n \text{ odd} \\ 2^{n-3}\exp(Q/2)\cosh(\kappa_n), & n \text{ even} \end{cases}$$

where $\lambda_n^2 = n-3$ and $\kappa_n^2 = 2n-6$. □

Corollary 5.2.3. *The simply connected 4-manifolds $Y(n)$ are not homotopy equivalent to any complex surface.*

Proof. If $Y(n)$ were homeomorphic to a complex surface, this computation shows that it would have to be minimal, since the formula for $\mathbf{D}_{Y(n)}$ does not contain a factor $\cosh(e)$ where $e^2 = -1$. Certainly the surface in question could not be elliptic since $c_1(Y(n))^2 \neq 0$. But neither could the surface be of general type since $Y(n)$ violates the Noether inequality. Thus $Y(n)$ is not homeomorphic to any complex surface.

D. Gomprecht [**Gt**] has computed the value of the Donaldson invariant $D_X(F^k)$ for any Horikawa surface X and k large, where F is the branched cover of the fiber f of F_{n-3}.

Certain of the Horikowa surfaces have two deformation types. Let $K(r)$ be the double cover of $\mathbf{F}_0$ branched over a smoothing of $6s_+ + 4rf$ and let $K'(r)$ be the double cover of $\mathbf{F}_{2r}$ branched over a disconnected branch locus which is a smoothing of $5s_+ \coprod s_-$. These are surfaces with $c_1^2 = 8r - 8$ and $c_2 = 40r - 4$ and Horikowa has shown that they are deformation inequivalent. If r is even these surfaces can be distinguished by the type of their intersection form. However, if r is odd, both these surfaces have odd intersection form and hence are homotopy equivalent.

Problem 20. Are the two deformation types of Horikowa surfaces diffeomorphic?

A related problem is

Problem 21. Are there restrictions on self-diffeomorphisms f of a minimal Kähler surface X (of nonnegative Kodaira dimension) with canonical class K_X beyond

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