PACIFIC JOURNAL OF MATHEMATICS
Vol. 194, No. 2, 2000

NONSYMPLECTIC 4-MANIFOLDS WITH ONE BASIC CLASS

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We produce examples of simply connected 4-manifolds which have (up to sign) only one class with a nontrivial Seiberg-Witten invariant. Furthermore, these manifolds admit no symplectic structure with either orientation.

1. Introduction.

In the past few years Zoltan Szabo [**S1**, **S2**] and the authors [**FS2**] have produced examples of simply connected irreducible 4-manifolds which do not admit a symplectic structure with either orientation. We shall call such manifolds *nonsymplectic*. Due to the nature of their construction, these manifolds have many basic classes. It is the purpose of this paper to construct families of examples of nonsymplectic 4-manifolds which (up to sign) have just one basic class.

The key to detecting that the manifolds of [**S1**, **S2**] and of [**FS2**] are not symplectic lies in the theorem of C. Taubes which states that the Seiberg-Witten invariant associated to the canonical class of a symplectic 4-manifold is ± 1. Recall that the Seiberg-Witten invariant SW_X of a smooth closed oriented 4-manifold X with $b^+ > 1$ is an integer valued function which is defined on the set of spin^c structures over X (cf. [**W**]). In case $H_1(X;\mathbf{Z})$ has no 2-torsion, there is a natural identification of the spin^c structures of X with the characteristic elements of $H^2(X;\mathbf{Z})$. In this case we view the Seiberg-Witten invariant as

$$\mathrm{SW}_X : \{k \in H^2(X,\mathbf{Z}) | k \equiv w_2(TX) \pmod 2\} \to \mathbf{Z}.$$

The Seiberg-Witten invariant SW_X is a smooth invariant whose sign depends on an orientation of $H^0(X;\mathbf{R}) \otimes \det H^2_+(X;\mathbf{R}) \otimes \det H^1(X;\mathbf{R})$. If $\mathrm{SW}_X(\beta) \neq 0$, then β is called a *basic class* of X. It is a fundamental fact that the set of basic classes is finite. If β is a basic class, then so is $-\beta$ with

$$\mathrm{SW}_X(-\beta) = (-1)^{(\mathrm{e}+\mathrm{sign})(X)/4}\,\mathrm{SW}_X(\beta)$$

where $\mathrm{e}(X)$ is the Euler number and $\mathrm{sign}(X)$ is the signature of X. Because of this, we shall say that X *has n basic classes* if the set $\{\beta \,|\, \mathrm{SW}_X(\beta) \neq 0\}/\{\pm 1\}$ consists of n elements.

There are abundant examples of 4-manifolds with one basic class. Minimal nonsingular algebraic surfaces of general type have one basic class (the canonical class) [**W**]. The authors have constructed many examples of minimal symplectic manifolds with one basic class and $\chi - 3 \leq {c_1}^2 < 2\chi - 6$, where $\chi = (b^+ + 1)/2$. (These manifolds cannot admit complex structures due to the geography of complex surfaces.) However, the examples produced below are the first nonsymplectic manifolds with one basic class.

As in [**FS2**] we need to view the Seiberg-Witten invariant as a Laurent polynomial. To do this, let $\{\pm\beta_1, \dots, \pm\beta_n\}$ be the set of nonzero basic classes for X. We may then view the Seiberg-Witten invariant of X as the 'symmetric' Laurent polynomial

$$\mathcal{SW}_X = b_0 + \sum_{j=1}^{n} b_j \left(t_j + (-1)^{(\mathrm{e}+\mathrm{sign})(X)/4}\, t_j^{-1} \right)$$

where $b_0 = \mathrm{SW}_X(0)$, $b_j = \mathrm{SW}_X(\beta_j)$ and $t_j = \exp(\beta_j)$. The examples of [**S1**, **S2**] and of [**FS2**] are obtained by producing 4-manifolds whose Seiberg-Witten Laurent polynomial $\mathcal{SW}_X$ has as a factor a nonmonic (symmetrized) Alexander polynomial of a knot or link. Taubes' result is then used to show that X cannot have a symplectic structure. It is not difficult to see that any nonsymplectic manifold (with $b^+ > 1$) which can be constructed by the techniques of [**FS2**] (as explained in [**FS2**], this includes the examples of Szabo) must have more than one basic class.

Whereas the examples of [**FS2**] are constructed by surgeries on embedded tori of self-intersection 0, the examples presented here arise from surgeries on higher genus surfaces. These examples are described in the next section.

2. A new family of 4-manifolds.

We begin by recalling the construction of [**FS2**]. Suppose that we are given a smooth simply connected oriented 4-manifold X containing an essential smoothly embedded torus T of self-intersection 0. Suppose further that $\pi_1(X \setminus T) = 1$ and that T is contained in a cusp neighborhood. Let $K \subset S^3$ be a smooth knot and M_K the 3-manifold obtained from 0-framed surgery on K. The meridional loop m to K defines a 1-dimensional homology class $[m]$ both in $S^3 \setminus K$ and in M_K. Denote by T_m the torus $S^1 \times m \subset S^1 \times M_K$. Then X_K is defined to be the fiber sum

$$X_K = X \#_{T=T_m} S^1 \times M_K = (X \setminus N(T)) \cup (S^1 \times (S^3 \setminus N(K))),$$

where $N(T) \cong D^2 \times T^2$ is a tubular neighborhood of T in X and $N(K)$ is a neighborhood of K in S^3. If λ denotes the longitude of K (λ bounds a surface in $S^3 \setminus K$) then the gluing of this fiber sum identifies $\{\mathrm{pt}\} \times \lambda$ with a normal circle to T in X. The main theorem of [**FS2**] asserts that X_K is homeomorphic to X, and

$$\mathcal{SW}_{X_K} = \mathcal{SW}_X \cdot \Delta_K(t)$$

where Δ_K is the symmetrized Alexander polynomial of K and $t = \exp(2[T])$.

To begin our construction, take X to be the $K3$-surface (which has $\mathcal{SW}_X = 1$) and let the torus T be a smooth fiber of an elliptic fibration on X. The pair (X, T) satisfies the hypotheses of [**FS2**]; so for any knot K we get a homotopy $K3$-surface X_K whose Seiberg-Witten invariant is $\mathcal{SW}_{X_K} = \Delta_K(t)$. The $K3$-surface, X, has a section (to the elliptic fibration) which is a smoothly embedded 2-sphere S of self-intersection -2, and $[S] \cdot [T] = 1$. The sum $[S] + [T]$ is represented by a smooth torus Σ with $[\Sigma]^2 = 0$ and $[\Sigma] \cdot [T] = 1$.

Suppose that the knot K has genus g. In the construction of X_K we have replaced a 2-disk in S (normal to T) with a punctured surface of genus g. Thus X_K contains a genus g surface S' satisfying $[S'] \cdot [S'] = -2$ and $[S'] \cdot [T] = 1$. Consider another smooth fiber T' of the elliptic fibration of $(X \setminus N(T)) \subset X_K$. Then $T' + S'$ is a singular curve with one double point, which can be smoothed to give an embedded surface Σ' of genus $g+1$ representing the homology class $[\Sigma'] = [T'] + [S']$. Thus $[\Sigma']^2 = 0$ and $[\Sigma'] \cdot [T] = 1$.

Next, let K' denote the left-handed trefoil knot in S^3. Since K' is a fibered genus 1 knot, the 4-manifold $S^1 \times M_{K'}$ is a smooth T^2-fiber bundle over T^2. Forming the fiber sum of $g+1$ copies of $S^1 \times M_{K'}$, we obtain

$$\begin{array}{ccccc} F = T^2 & \longrightarrow & Y & = & S^1 \times M_{K'} \#_F \cdots \#_F S^1 \times M_{K'} \\ & & \big\downarrow & & \\ & & C_0 & & \end{array}$$

where C_0 is a genus $g+1$ surface. Furthermore, $S^1 \times M_{K'}$ is a symplectic manifold [**Th**]. Notice that there is a section $C \subset Y$ of the fibration given by the connected sum of the $g+1$ tori T_{m_i}.

Generally, suppose that we are given symplectic 4-manifolds A and B and that $A \#_N B$ is their symplectic fiber sum along a symplectic torus of self-intersection 0. The adjunction formula implies that the canonical class K_A is orthogonal to $[N]$, as is K_B. The canonical class of $A \#_N B$ is then $K_{A \#_N B} = K_A + K_B + 2[N]$ (cf. [**G**]). Apply this fact to $X_{K'} = K3 \#_{T=T_m} S^1 \times M_{K'}$. Since $\mathcal{SW}_{X_{K'}} = \Delta_{K'}(t) = t - 1 + t^{-1}$ (where $t = \exp(2[T_m])$), we see that $K_{X_{K'}} = 2T_m = K_{K3} + K_{S^1 \times M_{K'}} + 2[T_m] = K_{S^1 \times M_{K'}} + 2[T_m]$. Hence $K_{S^1 \times M_{K'}} = 0$ and so $c_1(K_{S^1 \times M_{K'}}) = 0$. Now apply the fact g more times, this time fiber-summing along F. It follows that Y is a symplectic 4-manifold with $c_1(Y) = -2g[F]$. (Here, we identify $[F]$ with its Poincaré dual.)

Our example, corresponding to the genus g knot K is

$$Z_K = X_K \#_{\Sigma' = C} Y.$$

We perform this fiber sum so that Z_K is a spin 4-manifold [**G**].

Proposition 2.1. *The manifold Z_K is simply connected.*

Proof. The fundamental group $\pi_1(X_K \setminus \Sigma')$ is normally generated by a boundary circle of a normal disk to Σ'. Since $[\Sigma'] \cdot [T] = 1$, we may assume that this circle lies on a copy $\{\text{pt}\} \times T$ in the boundary $\partial D^2 \times T = \partial N(T)$. We claim that there are generators λ_1, λ_2, for $\pi_1(T)$ which bound vanishing cycles (disks of self-intersection -1) in $X \setminus (S \cup T')$. (Note that here we are identifying $X \setminus T$ with $X_K \setminus T$.) This claim can be seen to be true inside a $K3$ nucleus, i.e., in a regular neighborhood of the union of S and a cusp fiber. A Kirby calculus diagram for the nucleus is given in the Figure 1 below.

The section S is the union of the disk spanned by the circle labelled '-2' and the core of the 2-handle which is attached to it. The torus T is obtained as follows. The circle labelled '0' bounds a disk D which is punctured in two points by one of the dotted circles. Remove a pair of disks from D at these intersection points, and connect the boundaries of these disks with an annulus which surrounds the path γ in the diagram. The torus T is the union of the twice-punctured D and this annulus. We can see that the loop α of the diagram lies on T, and it is easy to deform β to also lie on T. (When this is done, α and β will intersect in a point.) Thus $H_1(T)$ is generated by the classes represented by the loops α and β. The vanishing cycles are the cores of the (-1)-framed 2-handles which are attached to α and β. This proves the claim.

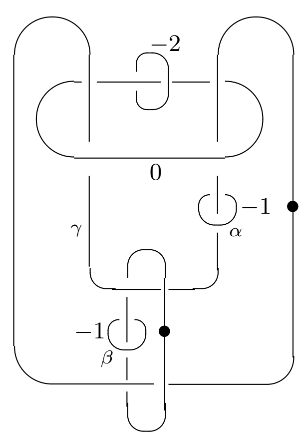

Figure 1.

This means that $\pi_1(\{\text{pt}\} \times T) \to \pi_1(X \setminus \Sigma)$ is the zero map; hence $\pi_1(\{\text{pt}\} \times T) \to \pi_1(X_K \setminus \Sigma')$ is also the zero map. However, $\pi_1(X_K \setminus \Sigma')$ is normally generated by the image of this map; so $X_K \setminus \Sigma'$ is simply connected.

Thus

$$\pi_1(Z_K) = \pi_1(X_K \setminus \Sigma') *_{\pi_1(C\times S^1)} \pi_1(Y \setminus C) = \pi_1(Y \setminus C)/\pi_1(C \times S^1).$$

Because $\pi_1(S^3 \setminus K')$ is normally generated by the meridian m, $\pi_1(S^1 \times (S^3 \setminus K')) = \pi_1((S^1 \times M_{K'})F)$ is normally generated by $\pi_1(S^1 \times m)$. An inductive application of Van Kampen's theorem shows that $\pi_1(Y)$ is normally generated by $\pi_1(S^1 \times m\# \cdots \# S^1 \times m) = \pi_1(C)$. Thus $\pi_1(Y \setminus C)$ is normally generated by $\pi_1(C \times S^1)$, and so $\pi_1(Z_K) = 1$. □

3. The Seiberg-Witten invariants of Z_K.

Consider first $H_2(Z_K)$. There is an important class $\tau \in H_2(Z_K)$ constructed as follows. In the construction of Z_K, the boundary of a tubular neighborhood $N(\Sigma')$ of Σ' in X_K is identified with the boundary of a tubular neighborhood $N(C)$ of C in Y. Fix a fiber F of Y, and let $F_0 = F\setminus(F\cap \mathrm{int}N(C))$. There is torus T'' which is a smooth fiber of the elliptic fibration of $X\setminus\{T\cup T'\} \subset X_K$ and such that if $T''_0 = T''\setminus(T''\cap \mathrm{int}N(\Sigma'))$, then $\partial T''_0 = \partial F_0$ in Z_K. Let τ denote the class $\tau = [T''_0 \cup F_0]$. Note that τ is represented by an embedded surface of genus 2, $\tau^2 = 0$, and $\tau \cdot [\Sigma'] = 1$. Then $H_2(Z_K)$ is generated by the image of $H_2(Y \setminus C)$, of $H_2(X_K \setminus \Sigma')$, and the class τ. The only other classes which could contribute to $H_2(Z_K)$ are the classes of rim tori, i.e., tori lying on $\partial N(\Sigma') = \partial N(C)$ in Z_K which have the form $\xi \times \partial D^2$ where D^2 is a normal disk to Σ' (or to C). The next lemma shows that in fact they are all trivial.

Lemma 3.1. *Each rim torus is homologically trivial in Z_K.*

Proof. A rim torus on $\partial N(C)$ has the form $\xi \times \partial D^2$, for some loop ξ on C. Recall that there is a fiber bundle $\varphi : Y \to C_0$ with fiber F. Let $Q = \varphi^{-1}(\xi) \subset Y$. We see that $\xi \times \partial D^2$ bounds the 3-chain $Q \setminus (\xi \times \mathrm{int}\, D^2)$ in $Y \setminus N(C) \subset Z_K$. □

Before we prove our main theorem, we recall that a 4-manifold W is said to have *simple type* if $\mathrm{SW}_W(k) \neq 0$ implies that

$$\dim \mathcal{M}_W(k) = \frac{1}{4}(k^2 - (3\,\mathrm{sign} + 2\,\mathrm{e})(W)) = 0$$

where $\mathcal{M}_W(k)$ is the Seiberg-Witten moduli space. Write the symmetrized Alexander polynomial of K as $\Delta_K(t) = a_0 + \sum_{n=1}^{d} a_n(t^n + t^{-n})$. We call d the *degree* of $\Delta_K(t)$. We are assuming that the genus of K is g; so $d \leq g$. If K is an alternating knot, for example, then $d = g$. Let us say that the Alexander polynomial of K has *maximal degree* if $d = g$.

Theorem 3.2. *Let K be a knot in S^3 whose Alexander polynomial has maximal degree. Then Z_K is of simple type and has* (*up to sign*) *a single basic* *class, $k = 2g\,\tau + 2[\Sigma']$. Furthermore, $|\mathrm{SW}_{Z_K}(k)| = a_d$, the top coefficient of $\Delta_K(t)$.*

Proof. Let U denote a nucleus in $X = K3$ which contains the fiber T and section S from the construction of X_K. We see that $(X \setminus U) \subset (X_K \setminus \Sigma')$. The homology $H_2(X \setminus U) \cong 2\,E_8 \oplus 2\,H$, where the negative definite E_8 forms are generated by the classes of embedded spheres of self-intersection -2, and the two hyperbolic pairs H are each generated by a torus T_i of self-intersection 0, and a sphere S_i of self-intersection -2 which meet transversely in a single point. The homology $H_2(X_K \setminus \Sigma')$ is generated by the image of $H_2(X \setminus U)$ together with $[\Sigma']$ and the classes of rim tori.

Next consider $Y = S^1 \times M_{K'} \#_F \cdots \#_F S^1 \times M_{K'}$ ($g+1$ copies) where F is the torus fiber of the fibration of $S^1 \times M_{K'}$ over the torus. Each $S^1 \times M_{K'}$ has the homology of $S^2 \times T^2$. Each fiber sum in the construction of Y increases the first betti number b_1 by 2 — the base of the fibration has genus increased by 1 — and H_2 is increased by the addition of two hyperbolic pairs as follows: Choose a standard basis $\{x_1, x_2\}$ for $H_1(F;\mathbf{Z})$. For example, x_1 is represented by a loop as shown in Figure 2. Each of the curves x_i bounds a punctured torus Γ_i in M_K. In Figure 3, x_1 is isotopic to the pictured curve x_1', and the punctured torus is composed of the twice-punctured disk which x_1' bounds, together with a 1-handle which pipes around the knot. Let x_i'' be a push-off of x_i in F. Then the linking number of x_i and x_i'' is $+1$. (Here we are using the fact that K' is a left-hand trefoil knot.) This means that the self-intersection number of Γ_i (as a surface in $M_K \times I$, say), keeping its boundary in F, is $+1$.

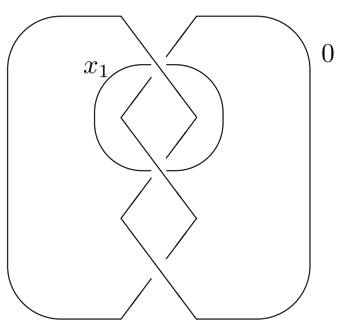

Figure 2.

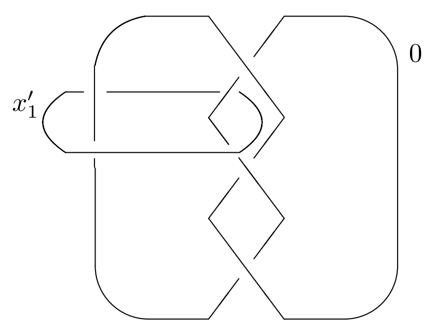

Figure 3.

Thus in $S^1 \times M_{K'} \#_F S^1 \times M_{K'}$ one produces genus 2 surfaces S_1', S_2', of self-intersection $+2$ which are formed from pairs of these tori. Let T_1', T_2' be the rim tori corresponding to x_2, x_1 (reversed on purpose). Then in $H_2(S^1 \times M_{K'} \#_F S^1 \times M_{K'};\mathbf{Z})$ two hyperbolic pairs are generated by the pairs $\{[S_i'], [T_i']\}$. Each further fiber sum adds two such hyperbolic pairs to

H_2. It follows that $H_2(Y \setminus C)$, is generated by the $[S_i']$, $[T_i']$, and the section class $[C]$.

Using our observations above, if k is a basic class of Z_K we can write

$$k = a\tau + b[\Sigma'] + \beta + \sum_{i=1}^{2} m_i[T_i] + n_i[S_i] + \sum_{j=1}^{2g} t_j[T_j'] + s_j[S_j']$$

where $a > 0$ and $\beta \in 2\,E_8 \subset H_2(X \setminus U)$. The adjunction inequality (see e.g. [**MST**]) states that if k is a basic class and B is an embedded surface of genus g_B and self-intersection $[B]^2 \geq 0$ then

$$2g_B - 2 \geq [B]^2 + |k \cdot [B]|.$$

In particular, this implies that if the self-intersection of B is $[B]^2 = 2g_B - 2$, then any basic class k must be orthogonal to it: $k \cdot [B] = 0$. Since T_i is a torus of self-interesection 0, it follows that $n_i = k \cdot [T_i] = 0$, and, since $[S_i] + [T_i]$ is also represented by an embedded torus of self-intersection 0, $m_i + n_i = k \cdot ([T_i] + [S_i]) = 0$. The same argument applies to show that $s_j = 0 = t_j$ for each $j = 1, \ldots, 2g$. Thus $k = a\tau + b[\Sigma'] + \beta$. Apply the adjunction inequality to the genus $g+1$ surface Σ' and the genus 2 surface representing τ to obtain:

$$(*) \qquad a = k \cdot [\Sigma'] \leq 2g, \qquad |b| = |k \cdot \tau| \leq 2.$$

Because k is a basic class, $\dim \mathcal{M}_{Z_K}(k) \geq 0$, hence

$$0 \leq k^2 - (3\,\mathrm{sign} + 2\,\mathrm{e})(Z_K).$$

Since $(3\,\mathrm{sign} + 2\,\mathrm{e})(X_K) = 0$, and $(3\,\mathrm{sign} + 2\,\mathrm{e})(Y) = 0$, it is easy to check that $(3\,\mathrm{sign} + 2\,\mathrm{e})(Z_K) = 8g$. Furthermore, β lies in the negative definite space $2\,E_8$; so if $\beta \neq 0$ then

$$0 \leq 2ab + \beta^2 - 8g < 2ab - 8g \leq 8g - 8g = 0.$$

This contradiction implies that $\beta = 0$; so $k = a\tau + b[\Sigma']$. Any of the (-2)-spheres generating the E_8's is orthogonal to k; hence orthogonal to each basic class of Z_K. It now follows from [**FS1**] that Z_K has simple type; $\dim \mathcal{M}_{Z_K}(k) = 0$. Thus we have

$$2ab = k^2 = (3\,\mathrm{sign} + 2\,\mathrm{e})(Z_K) = 8g.$$

It now follows from $(*)$ that $a = 2g$ and $b = 2$, as claimed.

Finally, we apply a theorem of Morgan, Szabo, and Taubes to calculate $\mathrm{SW}_{Z_K}(k)$. Since $k \cdot \Sigma' = 2g$, [**MST**] applies to give

$$\mathrm{SW}_{Z_K}(k) = \sum \mathrm{SW}_{Z_K}(k + 2[R]) = \pm \mathrm{SW}_{X_K}(2g\,T) \cdot \mathrm{SW}_Y(2g\,F)$$

where the the sum is taken for all distinct classes $k + 2[R]$ for R a rim torus. Thus the first equality follows from Lemma 3.1 which shows that each $[R] = 0$ in $H_2(Z_K; \mathbf{Z})$. Now Y is a symplectic manifold with $c_1(Y) = -2g\,F$;

so [**T**] implies that $\mathrm{SW}_Y(2g\,F) = \pm 1$. Since we are assuming that $g = d$, [**FS2**] implies that $\mathrm{SW}_{X_K}(2g\,T) = a_d$, and this completes the proof. □

We remark that in case the Alexander polynomial of the knot K does not have maximal degree, the above proof shows that $\mathcal{SW}_{Z_K} = 0$; this provides potential examples of simply connected irreducible 4-manifolds with trivial Seiberg-Witten invariants.

Corollary 3.3. *Let K be a knot in S^3 whose Alexander polynomial is not monic and has maximal degree. Then Z_K is a nonsymplectic 4-manifold with one basic class.*

Proof. The hypothesis implies that the only nontrivial Seiberg-Witten invariant of Z_K has value $\pm a_d \neq \pm 1$; so [**T**] implies that Z_K has no symplectic structure. Since Z_K contains an embedded sphere of self-intersection -2, neither does Z_K with its opposite orientation admit a symplectic structure. □

We close with a comment concerning the geography of our construction. If the genus of K is g then $c(Z_K) \equiv (3\,\mathrm{sign} + 2\,\mathrm{e})(Z_K) = 8g$ and $\chi(Z_K) \equiv \frac{b^+ + 1}{2}(Z_K) = g + 2$; so all these manifolds lie on the line of slope 8 passing through $(2,0) = (\chi(K3), c(K3))$ in the (χ, c)-plane. We could similarly perform our construction starting with $X = E(2n)$, the minimal elliptic surface without multiple fibers and with $\chi = 2n$. We then obtain the same result for the lattice points $(3n + g - 1, 8(g + n - 1))$, i.e., for the lattice points on the line of slope 8 through $(2n, 0) = (\chi(E(2n)), c(E(2n)))$.

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Received May 10, 1998 and revised December 15, 1998. The first author was partially supported NSF Grant DMS9704927 and the second author by NSF Grant DMS9626330.

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