Topology Vol. 14, pp. 257–269. Pergamon Press 1975. Printed in Great Britain

ON TOPOLOGICAL AND PIECEWISE LINEAR VECTOR FIELDS†

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(*Received* 31 *October* 1973; *revised* 15 *November* 1974)

INTRODUCTION

THE existence of a non-zero vector field on a differentiable manifold M yields geometric and algebraic information about M. For example,

(1) A non-zero vector field exists on M if and only if the tangent bundle of M splits off a trivial bundle.

(2) The kth Stiefel–Whitney class $W_k(M)$ of M is the primary obstruction to obtaining $(n-k+1)$ linearly independent non-zero vector fields on M [37; §39]. In particular, a non-zero vector field exists on a compact manifold M if and only if $X(M)$, the Euler characteristic of M, is zero.

(3) Every non-zero vector field on M is integrable.

Now suppose that M is a topological (TOP) or piecewise linear (PL) manifold. What is the appropriate definition of a TOP or PL vector field on M? If M were differentiable, then a non-zero vector field is just a non-zero cross-section of the tangent bundle $T(M)$ of M. In 1962 Milnor [29] defined the TOP tangent microbundle $\tau(M)$ of a TOP manifold M to be the microbundle $M \xrightarrow{\Delta} M \times M \xrightarrow{\pi} M$, where $\Delta(x) = (x, x)$ and $\pi(x, y) = x$. If M is a PL manifold, the PL tangent microbundle $\tau(M)$ is defined similarly if one works in the category of PL maps of polyhedra. If M is differentiable, then $\tau(M)$ is CAT ($CAT = TOP$ or PL) microbundle equivalent to $T(M)$. Kister [20], Kuiper and Lashoff [22], Mazur, and Hirsch then showed in 1965 that every CAT microbundle contained a unique $CAT\,R^n$-bundle, i.e. a bundle with fiber R^n and group the semi-simplicial group of CAT homeomorphisms of R^n keeping the origin fixed (see [11]). Thus, every CAT manifold M has a tangent $CAT\,R^n$-bundle $\tau(M)$.

Our first guess, then, at the appropriate definition of a non-zero CAT vector field on M is a non-zero CAT cross-section $s: M \to \tau(M)$. In 1965 R. F. Brown and E. Fadell ([3], [4]) showed that this was indeed a good definition. They showed that a non-zero CAT vector field exists on a compact CAT manifold M if and only if $X(M) = 0$. Actually, Brown and Fadell utilized the notion of the Nash tangent bundle $\bar{\tau}(M)$ of M [3]; however, it is easy to show that $\tau(M)$ has a non-zero cross section if and only if $\bar{\tau}(M)$ has a non-zero cross section.

The purpose of this paper is the further investigation of CAT vector fields on CAT manifolds. In particular, we find properties of CAT vector fields which are analogous to properties (1)–(3) given above. An important ingredient of the investigation is the result of R. Kirby and L. Siebenmann ([17], [18]) which states that if $n \geq 5$, then the stability map s: $(CAT_n, O_n) \to (CAT_{n+1}, O_{n+1})$ induces an isomorphism on the ith homotopy groups for $i \leq n+1$ and an epimorphism if $i = n+2$. When $CAT = PL$, Morris Hirsh, in an unpublished paper, proves the above result for $i \leq n$ and no restrictions on n. These results help us relate properties of vector bundles and $CAT\,R^n$-bundles.

The paper is divided into five sections. §0 establishes most of the notation and definitions to be used in later sections.

§1 deals with disk bundles. It is shown that any $CAT\,R^n$-bundle over a k-dimensional Euclidean neighborhood retract contains a CAT disk bundle if $k \leq n+2$ and $n \neq 4, 5$. Then, as a consequence of the work of Browder [2] and Hirsh [11], property (1) holds for CAT vector fields (and if $CAT = TOP$ the dimension of M is not four or five). We then discuss the dimension restrictions of the main theorem and show that in most cases they cannot be improved.

§2 concerns itself with property (2). The Stiefel–Whitney classes are defined for CAT

†This work represents a portion of the author's Ph.D. Thesis written under the direction of Professors Robert F. Brown and Robert D. Edwards at UCLA. The author would like to thank them for many helpful and encouraging discussions. The work was supported in part by NSF grant GP 36418X1 while at the Institute for Advanced Studies.

R^n-bundles according to Fadell [8], are axiomatized, and then given a geometrical interpretation similar to that for vector bundles. Motivated by this interpretation, the notion of $CAT\,k$-fields is then discussed in §3.

In §2 we also calculate some of the homotopy groups of the pair $(CAT_{n,k}, CAT_{n-k})$. §4 notes these are the coefficient groups for the obstruction to obtaining a CAT normal bundle to a locally flat embedding of a CAT manifold M^n in a CAT manifold Q^{n+q}. It is shown that if $n \leq q+1+j$ and $q \geq 5+j$, where $j = 0, 1, 2$, then M has a CAT normal bundle in Q. We then discuss the dimension restrictions and show that for every $q \geq 9$, the standard embedding of S^{q+3} in S^{2q+3} has two non-concordant CAT normal bundles. The section concludes with a discussion of non-existence of normal bundles in lower codimensions.

§5 deals with property (3). A CAT notion of integrability is defined for CAT vector fields, and then it is shown that, at least up to homotopy, all non-zero CAT vector fields are integrable.

Many of these results were announced in [38] and [39]. The author would also like to thank Larry Siebenmann for many helpful suggestions.

§0. NOTATION AND DEFINITIONS

A space X is a Euclidean neighborhood retract (ENR) if X is metrizable and there exists an integer $q > 0$ and an embedding $f\colon X \to R^q$ such that $f(X)$ has a neighborhood in R^q which retracts to $f(X)$.

A metrizable space X is said to be of dimension less than or equal to k, in symbols $\dim X \leq k$, if and only if every open covering α of X has a locally finite open refinement β such that its nerve N_β contains no simplex of dimension greater than k.

We use the fact that an ENR X with $\dim X \leq k$ is dominated by a simplicial complex which has no simplices of dimension greater than k [14]. In particular $H^i(X) = 0$ for all $i > k$.

We refer to [24] and [34] for the theory of semi-simplicial complexes and Δ-sets. In this paper we will rely heavily upon the homotopy groups of a *square*

$$\square \equiv \begin{array}{ccc} A & \supset & C \\ \cup & & \cup \\ B & \supset & D \end{array}$$

of pointed Δ-sets [32]. There are two exact homotopy sequences associated with $\square$, namely

$$\to \pi_{i+1}(\square) \to \pi_i(B, D) \to \pi_i(A, C) \to \pi_i(\square) \to$$
$$\to \pi_{i+1}(\square) \to \pi_i(C, D) \to \pi_i(A, B) \to \pi_i(\square) \to$$

We write the square $\square$ as $(A; C, B; D)$.

There is an exact sequence associated with the diagram

$$\begin{array}{ccccc} A & \supset & C & \supset & E \\ \cup & \square_3 & \cup & \square_1 & \cup \\ B & \supset & D & \supset & F \end{array}$$

namely

$$\to \pi_i(\square_1) \to \pi_i(\square_2) \to \pi_i(\square_3) \to$$

where $\square_2$ is the outside square.

Some Δ-groups

Let CAT represent either the category PL of piecewise linear manifolds and piecewise linear maps, or the category TOP of topological manifolds and continuous maps. As standard objects in the categories TOP and PL we have Euclidean n-space R^n, $D^n = \{x \in R^n \| x \| \leq 1\}$, and $S^n = \partial D^n$ is its boundary. R^n, D^n and S^n all have natural differentiable and TOP structures. Let $I^n = [-1, +1]^n$ and $\Sigma^n = \partial I^{n+1}$ and note that R^n, I^n and Σ^n have natural TOP and PL structures. We let Δ^n denote the standard n-simplex in R^{n+1}. There are natural inclusions $R^n \subset R^{n+k}$ and identifications $R^n \times R^k = R^{n+k}$. The symbol $\mathbf{O} \in R^n$ denotes the origin of R^n.

We now define several semi-simplicial and Δ-groups. Let $CAT = TOP$ or PL.

$CAT_{m,n}$ is the semi-simplicial group whose k-simplices are the set of fiber-preserving CAT homeomorphism $f\colon \Delta^k \times R^m \to \Delta^k \times R^m$ with $f = \text{identity}$ on $\Delta^k \times R^n \times \{\mathbf{0}\}$. We abbreviate $CAT_{m,0}$ by CAT_m.

$CAT_m(I)$ is the semi-simplicial complex whose k-simplices are the same as those of CAT_m with I^m replacing R^m.

$\widetilde{CAT}_{m,n}$ is the Δ-group whose k-simplices are the set of CAT homeomorphisms f: $\Delta^k \times R^m \to \Delta^k \times R^m$ with $f = \text{identity}$ on $\Delta^k \times R^n \times \{\mathbf{0}\}$, and if K is a subcomplex of Δ^k, $f^{-1}(K \times R^m) = K \times R^m$. We abbreviate $\widetilde{CAT}_{m,0}$ by $\widetilde{CAT}_m$.

O_n will denote the orthogonal group considered as a semi-simplicial complex [23].

There are natural inclusions

$$\begin{array}{ccccccc}
\widetilde{CAT}_{m,n} & \subset & \widetilde{CAT}_m & \subset & \widetilde{CAT}_{m+k} & \subset & \widetilde{CAT}_{m+k+q,q} \\
\cup & & \cup & & \cup & & \cup \\
CAT_{m,n} & \subset & CAT_m & \subset & CAT_{m+k} & \subset & CAT_{m+k+q,q} \\
 & & \cup & & \cup & & \\
 & & O_m & \subset & O_{m+k} & &
\end{array}$$

$$\text{and} \quad CAT_m(I) \subset CAT_m.$$

Here, as elsewhere in this paper, we shall not worry about writing PL_q or $\widetilde{PL}_q$ when strictly we should write PD_q or $\widetilde{PD}_q$. (see [23]).

If G is a semi-simplicial group and H is a semi-simplicial subgroup, then both G and H are Kan complexes and the natural projection $p\colon G \to G/H$, where G/H is the quotient complex, is a Kan fibration with fiber H. Thus, we have the exact homotopy sequence of this Kan fibration [24]

$$\cdots \to \pi_{i+1}(G/H) \to \pi_i(H) \to \pi_i(G) \to \pi_i(G/H) \to \cdots$$

We have the semi-simplicial fibrations [2], [11]

$$CAT_n \subset CAT_{n+1}(I) \xrightarrow{e} \Sigma^n$$

where e is evaluation at a fixed point $*$ of Σ^n. Here CAT_n is thought of as the CAT homeomorphisms of Σ^n keeping $*$ fixed (see [2]) and $CAT_{n+1}(I)$ as the CAT homeomorphisms of Σ^n (see [11]).

We will use the notions of CAT microbundle and CAT R^n-bundle interchangeably. This is justified by the coring theorems of [20] and [22].

If M is a CAT manifold, the CAT tangent bundle of M is denoted by $\tau(M)$. If M is a $DIFF$ manifold, the tangent vector bundle of M is denoted by $T(M)$. Note that when M is a $DIFF$ manifold, $\tau(M)$ can be regarded as $T(M)$ with group CAT_n [29].

Stability of CAT_n/O_n

Morlet [30, 31] and Kirby and Siebenmann [17], [18], using only the sliced classification theorem for manifolds with boundary, show that

$$\pi_i(\mathrm{Aut}_{DIFF}(D^{q-1} \times I \text{ rel}])) \cong \pi_{i+q+1}(CAT_q\,;\, O_q, CAT_{q-1};\, O_{q-1})$$

for $q \geq 6$, $i \geq 0$. On the left is the ith homotopy group of the semi-simplicial space of $DIFF$ automorphisms of $D^q \times I$ fixing $] = (\partial D^q \times I) \cup (D^q \times 1)$, which is just the ith homotopy group of the space of $DIFF$ pseudo-isotopies of D^q. In [30] and [31] Morlet announces and in an unpublished paper, Hirsch establishes the same isomorphism when $CAT = PL$ and no restrictions on q. Using this result, along with Cerf's pseudo isotopy theorem for the disk and the s-cobordism, Kirby and Siebenmann [17], [18], Morlet [30], [31], and Hirsch establish

THEOREM 0.1. *If $q \neq 4, 5$, then $\pi_i(CAT_q\,;\, O_q, CAT_{q-1};\, O_{q-1}) = 0$ for $i \leq q+1$, or for all i and $q \leq 2$.*

THEOREM 0.2. *$\pi_i(PL_q;\ O_q,\ PL_{q-1};\ O_{q-1}) = 0$ for $i \leq q$.*

Recently I. Volodin and A. Hatcher have computed $\pi_1(\mathrm{Aut}_{DIFF}(D^q))$ and defined Whitehead groups $W^0(G)$, $W^1(G), \ldots$ for any group G, which imply (see [44]),

THEOREM 0.3. *If $q \geq 8$, then $\pi_{q+2}(CAT_q;\ O_q,\ CAT_{q-1};\ O_{q-1}) = Z_2 \oplus \mathrm{Wh}^3(0)$.*

Definition of a *CAT* vector field

As motivated in the introduction we define a *CAT* vector field as follows.

Definition 0.4. A non-zero *CAT* vector field on a *CAT* manifold M is a non-zero *CAT* cross-section $s: M \to \tau(M)$. Two *CAT* vector fields, s_0 and s_1, are *homotopic* if there exists a *CAT* map $S: I \times M \to \tau(M)$ such that $s_t = S(t, \cdot)$ is a non-zero *CAT* vector field for all $t \in I$.

§1. DISK BUNDLES

A vector bundle ξ enjoys the property that ξ splits off a trivial bundle if and only if ξ has a non-zero cross-section. In 1965 Browder [2] and Hirsch [11] showed that a *CAT* R^n-bundle ξ with a non-zero *CAT* cross-section splits off a trivial bundle if and only if ξ contains a *CAT* disk bundle. In particular, the *CAT* tangent bundle $\tau(M)$ of a *CAT* manifold which has a non-zero *CAT* vector field splits off a trivial bundle if and only if $\tau(M)$ contains a *CAT* disk bundle. However, one easily observes that there exists a *CAT* R^n-bundle which either contains no *CAT* disk bundle or a *CAT* R^n-bundle containing two inequivalent *CAT* disk bundles (both occur, see Browder [2]), for consider the communative diagram

$$\begin{array}{ccccc} CAT_n & \subset & CAT_{n+1}(I) & \to & \Sigma^n \\ \cup & & \cup & & \\ O_n & \subset & O_{n+1} & \to & S^n \end{array} \tag{1.0}$$

where the maps onto S^n and Σ^n are evaluation and the rows are fibrations (see §0). If every *CAT* R^n-bundle contained a unique *CAT* disk bundle, then $\pi_*(CAT_n,\ CAT_n(I)) = 0$. But by (1.0) $\pi_*(CAT_{n+1}(I),\ O_{n+1}) \cong \pi_*(CAT_n,\ O_n)$ and thus all these groups must be zero by induction on n, which is well known to be false.

In this section we will investigate the relationship between the stability of CAT_n/O_n and disk bundles.

THEOREM 1.1. $\pi_*(CAT_{q+1},\ CAT_{q+1}(I)) \cong \pi_*(CAT_{q+1};\ CAT_q,\ O_{q+1};\ O_q)$

Proof. Consider the communative diagram

$$\begin{array}{ccccc} CAT_{q+1} & \supset & CAT_{q+1}(I) & \supset & CAT_q \\ \cup\ \square_3 & & \cup\ \square_1 & & \cup \\ O_{q+1} & \supset & O_{q+1} & \supset & O_q \end{array}$$

and its associated exact sequence

$$\to \pi_i(\square_1) \to \pi_i(\square_2) \to \pi_i(\square_3) \to$$

where $\square_2$ is the outside square. By (1.0) $\pi_*(\square_1) = 0$. The exact homotopy sequences associated with the square $\square_3$ yields that $\pi_*(\square_3) \cong \pi_*(CAT_{q+1},\ CAT_{q+1}(I))$ and thus that

$$\pi_*(\square_2) = \pi_*(CAT_{q+1};\ O_{q+1},\ CAT_q;\ O_q) \cong \pi_*(CAT_{q+1},\ CAT_{q+1}(I)).$$

Theorems 0.1, 0.2 and 1.1 immediately imply

COROLLARY 1.2. *$\pi_i(CAT_n,\ CAT_n(I)) = 0$ if either*

(i) *$i \leq n+1$ and $n \neq 4, 5$*

(ii) *$i \leq n$ and $CAT = PL$, or*

(iii) *i arbitrary and $n \leq 2$.*

COROLLARY 1.3. *Let X be an ENR, $\dim X \leq k$. Then any CAT R^n-bundle over X contains a*

CAT disk bundle if either

(i) $k \leq n+2$ *and* $n \neq 4, 5$,

(ii) $CAT = PL$ *and* $k \leq n+1$, *or*

(iii) k *arbitrary and* $n \leq 2$.

It is uniquely determined (up to *CAT* isomorphism) *if either*

(i) $k \leq n+1$ *and* $n \neq 4, 5$

(ii) $CAT = PL$ *and* $k \leq n$, *or*

(iii) k *arbitrary and* $n \leq 2$.

Remark. For $CAT = TOP$, Adachi[1] has obtained Corollary 1.3 for the special case that $k \leq n-3$ and $n \geq 6$, while Rourke and Sanderson[33] have also obtained Corollary 1.3 for the case that $k \leq n-1$ and $n \geq 6$. For $CAT = PL$, Hirsch[12] obtained Corollary 1.3 when $k \leq n$.

Theorems 0.3 and 1.1 establish that for $n \geq 8$, Corollary 1.2 and the uniqueness part of Corollary 1.3 are the best possible, namely

COROLLARY 1.4 *If* $n \geq 8$, *then* $\pi_{n+2}(CAT_n, CAT_n(I)) = Z_2 \oplus \mathrm{Wh}^3(O) \neq 0$.

COROLLARY 1.5. *A CAT n-manifold* ($n \neq 4, 5$ *if* $CAT = TOP$) *has a unique CAT tangent disk bundle. Consequently, M has a non-zero CAT vector field if and only if its tangent bundle splits off a trivial bundle.*

Anomalies in lower dimensions

Consider the communative diagram

$$\begin{array}{ccccc} TOP_4 & \supset & TOP_4(I) & \supset & TOP_3 \\ \cup\ \square_3 & & \cup\ \square_1 & & \cup \\ PL_4 & \supset & PL_4(I) & \supset & PL_3 \end{array}$$

and its associated exact sequence of squares

$$\rightarrow \pi_i(\square_1) \rightarrow \pi_i(\square_2) \rightarrow \pi_i(\square_3) \rightarrow$$

where $\square_2$ is the outside square. From the commutative diagram

$$\begin{array}{ccccc} TOP_3 & \subset & TOP_4(I) & \rightarrow & \Sigma^3 \\ \cup & & \cup & & \\ PL_3 & \subset & PL_4(I) & \rightarrow & \Sigma^3 \end{array}$$

with rows that are fibrations we deduce that $\pi_*(\square_1) = 0$, hence $\pi_*(\square_2) \cong \pi_*(\square_3)$. But Corollary 1.2 (ii) implies that $\pi_i(\square_3) = \pi_i(TOP_4(I))$ for $i \leq 4$. Also, Morlet[31] and Kirby and Siebenmann[18] have shown that $\pi_*(TOP_3, PL_3) = 0$ so that $\pi_*(\square_2) \cong \pi_*(TOP_4, PL_4)$, hence $\pi_i(TOP_4, TOP_4(I)) \cong \pi_i(TOP_4, PL_4)$ for $i \leq 4$.

Kirby[16] has conjectured that $\pi_3(TOP_4, PL_4) = Z_2$. So we conjecture

CONJECTURE. *There exists a TOP* R^4*-bundle over a 4-complex which contains no disk bundle.*

QUESTION. Does the tangent bundle of a *TOP* 4 or 5 manifold contain a disk bundle?

§2. STIEFEL-WHITNEY CLASSES FOR *CAT* R^n-BUNDLES

If M is a *DIFF* n-manifold, the qth Stiefel–Whitney class of M, $W_q(M)$, was classically defined as the primary obstruction to finding $n-q+1$ linearly independent vector fields on M. $W_q(M)$ is an element of $H^q(M; Z)$ for q odd or $q = n$ and an element of $H^q(M; Z_2)$ for q even and $q < n$; in the first case we use twisted coefficients (see §38 of [37]). In 1950, Wu[43], basing his work on that of Thom[41], derived a formula for the mod 2 Stiefel–Whitney classes which only depended upon the cohomology ring of M and the Steenrod squaring operations. This allowed one to define the mod 2 Stiefel–Whitney classes for spherical fibrations. However, the realization of these Stiefel–Whitney classes as obstructions was no longer obvious. In this section we recover such a realization for almost all the Stiefel–Whitney classes.

We will now let $CAT = DIFF$, PL, TOP or G where by a G R^n-bundle we mean the disk

fibration associated with a spherical fibration; and by a *DIFF* R^n-bundle we mean an n-plane bundle.

Let $\xi = (E, p, B)$ be a *CAT* R^n-bundle over a space B. We have the following diagram

$$\begin{array}{ccc} H^n(E, E_0; Z_2) & \xleftarrow{\Phi} & H^0(B; Z_2) \\ \Big\downarrow{\scriptstyle \mathrm{Sq}^i} & & \\ H^{n+i}(E, E_0; Z_2) & \xleftarrow{\Phi} & H^i(B; Z_2) \end{array}$$

where Φ is the Thom isomorphism[8], Sq^i is the ith Steenrod square[36], and $E_0 = E - (\text{zero section})$. The ith Stiefel–Whitney class of ξ, $w_i(\xi)$, is defined by

$$w_i(\xi) = \Phi^{-1}\mathrm{Sq}^i\Phi(1).$$

where 1 is the unit of the cohomology ring $H^*(B; Z_2)$.

Axioms for the *CAT* Stiefel–Whitney classes of *CAT* R^n-bundles

(1) To each *CAT* R^n-bundle ξ over a space B having the homotopy type of a *CW* complex there corresponds an element $W(\xi) = 1 + w_1(\xi) + \cdots + w_n(\xi)$ of $H^*(B; Z_2)$, where $w_i(\xi) \in H^i(B; Z_2)$.

(2) For a *CAT* bundle map $f = (f_E, f_B)$: $\xi \to \eta$, $f_B^*(W(\eta)) = W(\xi)$.

(3). If $\xi = n^q \oplus \epsilon^{n-q}$ where η^q is a *CAT* R^q-bundle over B and ϵ^{n-q} is a trivial R^{n-q}-bundle over B, then $W(\xi) = W(\eta)$.

(4) For each n there exists a *CAT* R^n-bundle ξ such that $w_n(\xi) \neq 0$.

The Stiefel–Whitney classes defined above satisfy axioms (1)–(4) for *CAT* = *DIFF*, *PL*, *TOP* or *G*. We now show that these are the only classes satisfying the axioms, i.e.

THEOREM 2.0. *Suppose* $\{\tilde{W}\}$ *is another collection of classes satisfying axioms* (1)–(3). *If* $\xi = (E, p, B)$ *is a CAT* R^n*-bundle over B and* $q \neq 4, 5$ *if CAT = TOP, then* $\tilde{w}_q(\xi) = \lambda w_q(\xi)$ *where* $\lambda \in H^0(B; Z_2)$ *depends only on q. If* $\{\tilde{W}\}$ *also satisfies axiom* (4) *and* ξ *and q are as above, then* $\tilde{w}_q(\xi) = w_q(\xi)$.

Remark. Note that for *CAT* = *DIFF* these axioms are different than the usual ones[28]. Axiom (3) is weakened at the expense of strengthening axiom (4).

Proof of Theorem 2.0. There exists a *CW* complex $BCAT_n$ and a *CAT* R^n-bundle γ_n over $BCAT_n$ that is universal in the following sense. Let $\xi = (E, p, B)$ be a *CAT* R^n-bundle over a space B having the homotopy type of a *CW* complex. Then there exists a map f: $B \to BCAT_n$ such that the pull-back bundle $f^*(\gamma_n)$ is *CAT* bundle isomorphic to ξ and f is unique up to homotopy.

Let $B^{(q)}$ denote the q-skeleton of $BCAT_n$. Since $\pi_i(CAT_n, CAT_q) = 0$ for $i < q$, where if *CAT* = *TOP* we require $n, q \neq 4$ (for *CAT* = *G* see [32], *CAT* = *DIFF* see [37], *CAT* = *PL* see [10], *CAT* = *TOP* see [17]), $\gamma_n|B^{(q)}$ is *CAT* bundle isomorphic to $\eta^q \oplus \epsilon^{n-q}$, where η^q is a *CAT* R^q-bundle over $B^{(q)}$ and ϵ^{n-q} is a trivial *CAT* R^{n-q}-bundle over $B^{(q)}$. By axioms (2) and (3), $i_q^*\tilde{w}_q(\gamma_n) = \tilde{w}_q(\eta^q)$, where i^* is the cohomology monomorphism induced by the inclusion i: $B^{(q)} \to BCAT_n$. Also $i_q^* w_q(\gamma_n) = w_q(\eta^q)$. Let E^q be the total space of η^q, $E_0{}^q = E^q$-(zero-section), $\bar{p}$: $E^q \to B^{(q)}$ the projection map, and $\bar{p}_0 = p|E_0{}^q$. Let $\bar{\eta}$ be the bundle over $E_0{}^q$ induced by $\bar{p}_0$, i.e. $\bar{\eta} = (\bar{p}_0)^*(\eta^q)$. If *CAT* = *TOP* or *PL*, Corollary 1.2 says that η^q, hence $\bar{\eta}$, contains a disk bundle so that by the theorem of Browder and Hirsch mentioned in §i, $\bar{\eta}$ is *CAT* bundle isomorphic to $\eta^{q-1} \oplus \epsilon^1$ where η^{q-1} is a *CAT* R^{q-1}-bundle. This is also clearly true for *CAT* = *DIFF* and is true for *CAT* = *G* by virtue of the fact that $\pi_i(F_q, G_q) = 0$ for $i \leq 2q - 3$, where F_q = space of base point preserving homotopy equivalences $S^q \to S^q$ and G_q = space of homotopy equivalences $S^{q-1} \to S^{q-1}$. Thus, $0 = \tilde{w}_q(\bar{\eta}) = \tilde{w}_q(\bar{p}_0^*(\eta^q)) = \bar{p}_0^*(\tilde{w}_q(\eta^q))$ by axioms (2) and (3). We have the following Gysin sequence[36],

$$\to H^0(B^{(q)}; Z_2) \xrightarrow{\cup w_q(\eta^q)} H^q(B^{(q)}; Z_2) \xrightarrow{\bar{p}_0^*} H^q(E_0{}^q; Z_2) \to$$

By exactness of this sequence, $\tilde{w}_q(\eta^q) = \lambda w_q(\eta^q)$ where $\lambda \in H^0(B^{(q)}; Z_2)$. Thus $i_q^*(\tilde{w}(\gamma_n)) = i_q^*(\lambda w_q(\gamma_n))$ so that $\tilde{w}(\gamma_n) = \lambda w_q(\gamma_n)$. Let $f\colon B \to BCAT_n$ classify ξ. Then axiom (2) yields $\tilde{w}_q(\xi) = f^*(\tilde{w}_q(\gamma_n)) = \lambda f^*(w_q(\gamma_n)) = \lambda w_q(\xi)$, $\lambda \in H^0(B; Z_2)$. The integer λ only depends upon q, not on ξ. The theorem now follows.

Remark. Note that the same proof also shows that axioms (1)–(4) uniquely determine Stiefel–Whitney classes for *PL* and *TOP* block bundles.

Now we let $CAT = TOP$ or PL.

CAT-Stiefel manifolds

The classical Stiefel manifold $V_{n,k}$ is defined as the quotient O_n/O_{n-k}. We now define analogous Stiefel manifolds for the topological and piecewise linear categories.

Definition 2.1. (a) The *CAT–Stiefel manifold* ${}^0V_{n,k}^{CAT}$ is the quotient complex $CAT_n/CAT_{n,k}$, i.e. the quotient complex of CAT_n under the equivalence relation $\sigma_0 \sim \sigma_1$ if $\sigma_0^{-1}\sigma_1$ is a k-simplex of $CAT_{n,k}$, where σ_0 and σ_1 are k-simplices of CAT_n.

(b) The *CAT–Stiefel manifold* ${}^1V_{n,k}^{CAT}$ is the quotient complex CAT_n/CAT_{n-k}, i.e. the quotient complex of CAT_n under the equivalence relation $\sigma_0 \sim \sigma_1$ if $\sigma_0^{-1}\sigma_1$ is a simplex of CAT_{n-k}.

The group CAT_n acts naturally on the right of ${}^tV_{n,k}^{CAT}$, $t = 0, 1$, so if ξ is a $CAT\,R^n$-bundle over a simplicial complex X, we can associate to ξ the semi-simplicial bundles ${}^tV_{n,k}^{CAT}(\xi)$, with fiber ${}^tV_{n,k}^{CAT}$, $t = 0, 1$. The reason for defining two CAT–Stiefel manifolds is explained by

PROPOSITION 2.2. (a) *There is a cross-section* $s\colon X \to {}^0V_{n,k}^{CAT}(\xi)$ $[{}^1V_{n,k}^{CAT}(\xi)]$ *if and only if* ξ *contains* [splits off] *a trivial* k*-dimensional* CAT *sub-bundle.*

(b) *There exists integers* k, n *and a* $CAT\,R^n$*-bundle* ξ *over a complex* X *such that* ξ *contains a trivial* k*-dimensional* CAT *sub-bundle but does not split off this sub-bundle.*

The proof of 2.2(a) is clear. In §4 we remark that 2.2(b) is equivalent to saying that there exists a CAT locally flat imbedding $M^n \to N^{n+q}$ of a CAT manifold M in a CAT manifold Q with no CAT normal bundle. Such imbeddings are known to exist[12], [13].

Our first goal is to calculate some of the homotopy groups of ${}^tV_{n,k}^{CAT}$, $t = 0, 1$, using the stability results of Kirby and Siebenmann and Morlet (Theorem 0.1), Hirsch (Theorem 0.2) and results of Rourke and Sanderson[32], [33] and Millett[25].

Note that there is a natural inclusion $j\quad V_{n,k} \to {}^1V_{n,k}^{CAT}$ and that $\pi_*({}^1V_{n,k}^{CAT}, V_{n,k}) \cong \pi_*(CAT_n; O_n, CAT_{n-k}; O_{n-k})$.

THEOREM 2.3. *Let* $j\colon V_{n,k} \to {}^1V_{n,k}^{CAT}$ *be the natural inclusion. If* $n-k+i \geq 5$ *or if* $n-k \geq 4$ *and* $CAT = PL$, *then* $j_*\colon \pi_i(V_{n,k}) \to \pi_i({}^1V_{n,k}^{CAT})$ *is a monomorphism with left inverse for* $i \leq 2(n-k)-3$, *and* j_* *is an isomorphism if either*

(i) $i \leq n-k+1+j$ *and* $n-k \geq 5+j$, $j = 0, 1$

(ii) $CAT = PL$ *and* $i \leq n-k+1$

(iii) $n \leq 2$ *and* i *is arbitrary*

(iv) $i \leq 3$ *and* $n = 3$, *or*

(v) $i \leq 1$, $n \geq 5$, *and* $n-k \leq 3$

Proof. Consider the triad $T = (CAT_n; O_n, CAT_{n-k}; O_{n-k})$ and the following communative diagram with exact row and column

$$\begin{array}{ccccccccc}
 & & & & & & \downarrow & & \\
 & & & & & & \pi_{i+1}(T) & & \\
 & & & & & & \downarrow & & \\
 & & & & & & \pi_i(O_n, O_{n-k}) & & \\
 & & & & & & {\scriptstyle i^*}\swarrow & & \\
 & & & & \pi_i(\widetilde{CAT}_n, \widetilde{CAT}_{n-k}) & & \downarrow j_* & & \\
 & & & & & & {\scriptstyle k^*}\nwarrow & & \\
 & & & & & & \pi_i(CAT_n, CAT_{n-k}) & & \\
 & & & & & & \downarrow & & \\
\to \pi_{i+1}(T) & \to & \pi_i(CAT_{n-k}, O_{n-k}) & \xrightarrow{s_*} & & & \pi_i(CAT_n, O_n) & \to & \pi_i(T) \to \\
 & & & & & & \downarrow & &
\end{array} \tag{2.4}$$

where s_*, l_*, j_*, and k_* are induced by the natural inclusion maps. Rourke and Sanderson have shown that l_* is an isomorphism for $i < 2(n-k)-3$ if $CAT = PL$ [32] and if $CAT = TOP$ we further require that $n-k+i \geq 5$ [32]. Thus, the theorem now follows from Theorems 0.1, 0.2 and the commutativity of (2.4).

THEOREM 2.5. *Let* j: $V_{n,k} \to {}^0V_{n,k}^{CAT}$ *be the natural inclusion and suppose* $n-k \geq 3$. *Then* j_*: $\pi_i(V_{n,k}) \to \pi_i({}^0V_{n,k}^{CAT})$ *is an isomorphism for* $i < 2(n-k)-3$ *and an epimorphism for* $i = 2(n-k)-3$.

Proof. In [25], Millett studies the PL-Stiefel manifold ${}^0V_{n,k}^{PL}$ and shows that, when $n-k \geq 3$, j_*: $\pi_i(V_{n,k}) \to \pi_i({}^0V_{n,k}^{PL})$ is an isomorphism for $i < 2(n-k)-3$ and an epimorphism for $i = 2(n-k)-3$. Now ${}^0V_{n,k}^{CAT}$ can be identified with the semi-simplicial space $\mathscr{E}_{CAT}(S^k, S^n, i|S^0)$ of CAT locally flat embeddings of S^k in S^n which restrict to the identity map on S^0, i.e. a typical p-simplex of $\mathscr{E}_{CAT}(S^k, S^n; i|S^0)$ is a p-cell F: $\Delta^p \times S^k \to \Delta^p \times S^n$ of CAT locally flat imbeddings such that $F|\Delta^p \times S^0 =$ identity of $\Delta^p \times S^0$. Miller [27] has proven a sliced version of his approximation theorem [26] which implies that if $n-k \geq 4$ $\pi_*(\mathscr{E}_{TOP}(S^k, S^n; i|S^0), \mathscr{E}_{PL}(S^k, S^n; i|S^0)) = 0$. By developing a sliced theory for putting a PL structure on a TOP manifold relative to a submanifold, one can prove a sliced version of Miller's approximation theorem with domain a *manifold* (rather than an arbitrary polyhedron) which implies that if $n-k \geq 3$ $\pi_*(\mathscr{E}_{TOP}(S^k, S^n); i|S^0), \mathscr{E}_{PL}(S^k, S^n; i|S^0)) = 0$. See [40] for details. The theorem now follows.

The Stiefel-Whitney classes as obstructions

As the proof of Theorem 2.0 we let $BCAT_n$ denote the classifying space for $CAT\,R^n$-bundle. There exists a semi-simplicial Kan set $BCAT'_n$ and a semi-simplicial CAT bundle γ_n^{CAT} over $BCAT'_n$ which is universal and such that the geometric realization of $BCAT'_n$ is $BCAT_n$ [17]. Let ${}^tc_i^{CAT}(n)$ be the primary obstruction to obtaining a cross-section s: $BCAT'_n \to {}^tV_{n,n-i+1}^{CAT}(\gamma_n^{CAT})$.

Let $\xi = (E, p, B)$ be a $CAT\,R^n$-bundle over an $ENR\,B$ and let f: $B \to BCAT_n$ classify ξ. We define the *primary obstruction to reducing the group of* ξ *from* CAT_n *to* $CAT_{n,n-i+1}$ (respectively CAT_{i-1}) to be that cohomology class ${}^0c_i^{CAT}(\xi)$ (respectively ${}^1c_i^{CAT}(\xi)$) such that ${}^tc_i^{CAT}(\xi) = f^*({}^tc_i^{CAT}(n))$, $t = 0, 1$, where f^* is the homomorphism on cohomology induced by f. As a consequence of Theorems 2.3 and 2.5 we have that

THEOREM 2.6. *The class* ${}^tc_q^{CAT}(\xi)$ *is an element of* $H^q(M; Z)$ *for* q *odd or* $q = n$ *and an element of* $H^q(M; Z_2)$ *for* q *even and* $q < n$ (in the first case we use twisted coefficients) *if any of the following conditions hold*

(i) $t = 0$ *and* $q \geq 5$

(ii) $t = 1$ *and* $CAT = PL$

(iii) $t = 1$, $n \geq 5$ *and* $q \neq 5, 6$, *or*

(iv) $t = 1$ *and* $n \leq 3$.

Since we wish to compare $w_q(\xi)$, which is an element of $H^q(B; Z_2)$, with ${}^tc_q^{CAT}(\xi)$, we reduce ${}^tc_q^{CAT}(\xi)$ mod 2, denoted ${}^tc_q^{CAT}(\xi)_2$.

We are now in a position to prove the main theorem of this section, namely.

THEOREM 2.7. *Let* $\xi = (E, p, B)$ *be a* $CAT\,R^n$*-bundle over an* $ENR\,B$. *If* $q \leq n$ *and* (t, q, n) *satisfy any of the conditions* (i)–(iv) *of Theorem* 2.6, *then* $w_q(\xi) = {}^tc_q^{CAT}(\xi)_2$.

Proof. We abbreviate ${}^tc_q^{CAT}(\xi)$ by $c_q(\xi)$. It follows from the proof of Theorem 2.0 that if we can show that for every $CAT\,R^n$-bundle ξ over B, the classes $C(\xi) = \Sigma c_q(\xi)$, the sum being taken over all q satisfying the hypothesis of the theorem, satisfy axioms (2)–(4) for the CAT Stiefel–Whitney classes, then $c_q(\xi) = w_q(\xi)$ for all q satisfying the hypothesis of the theorem.

Axiom (2) is clearly satisfied. For axiom (3) suppose $\xi^n = \eta^q \oplus \epsilon^{n-q}$ for some q satisfying the hypothesis of the theorem. Let f: $B \to BCAT_n$ and g: $B \to CAT_q$ classify ξ and η, respectively. Let p: $BCAT_q \to BCAT_n$ be the natural map. Then

$$c_i(\xi) = f^*(c_i(n)) = (pg)^*c_i(n) = g^*c_i(q) = c_i(\eta).$$

Finally, for any n, let $\xi = (E, p, B)$ be the CAT R^n-bundle defined as follows. Let $B = RP^n =$ real projective space and let π: $S^n \to RP^n$ be the usual quotient map. Define p: $E \to RP^n$ to be that vector bundle such that $p^{-1}(\pi^{-1}(x))$ is the set of all vectors orthogonal to x, considered as a vector in R^{n+1}. We can define a non-zero cross-section to this bundle everywhere

except at the north and south poles. This singularity corresponds to a generator of $\pi_n(S^n)$. Hence, as $c_n(\xi)$ is the only obstruction to extending this non-zero cross-section, $(c_n(\xi))_2 \neq 0$ (see [29]). Thus axiom (4) is satisfied.

§3. *CAT k*-FIELDS

In §2 we identified the Stiefel–Whitney class of a *CAT* R^n-bundle ξ as the primary obstruction (reduced mod 2) to reducing the group of ξ. If M is a differentiable n-manifold, then the group of $T(M)$ can be reduced from O_n to O_{n-k} if and only if there exists a k-field on M, that is, k-linearly independent vector fields on M. Our goal in this section is to define a notion of a *CAT* k-field on a *CAT* manifold M and show that one exists if and only if the group of $\tau(M)$ can be reduced from CAT_n to $CAT_{n,k}$. This then answers a question posed by Fadell in [7].

A first guess at what a *CAT* k-field on a *CAT* manifold M should be is just k-linearly independent non-zero cross-sections to $\tau(M)$. But linear independence makes no sense in *CAT*, since CAT_n does not preserve it. We will show in Theorem 3.2 that the following definition is exactly what we are looking for.

Definition 3.1. A *CAT k-field* on a *CAT* manifold M is a map (semi-simplicial if $CAT = PL$) $p\colon M \to \mathscr{E}_{CAT}(R^k, M)$ such that for $b \in M$ $p(b)(\mathbf{O}) = b$. Two *CAT* k-fields, p_0, and p_1, are *homotopic* if there exists a map $P\colon I \times M \to \mathscr{E}_{CAT}(R^k, M)$ such that $p_t = P(t, \cdot)$ is a *CAT* k-field for all $t \in I$.

Here $\mathscr{E}_{TOP}(R^k, M)$ is the space of all locally flat embeddings of R^k into M, and $\mathscr{E}_{PL}(R^k, M)$ is the semi-simplicial set of *PL* locally flat embeddings of R^k into M of which a typical k-simplex is a k-cell $H\colon \Delta^k \times R^k \to \Delta^k \times M$ of *PL* locally flat embedding of R^k into M.

THEOREM 3.2. *Let M be a closed CAT n-manifold. If $n - k \geq 3$ and $n \geq 5$, then there is a* 1–1 *correspondence between homotopy classes of CAT k-fields on M and homotopy classes of reductions of the group of $\tau(M^n)$ from CAT_n to $CAT_{n,k}$.*

Proof. Let $CAT\,\mathscr{R}_k(M)$ denote the set of homotopy classes of *CAT* reductions of the group of $\tau(M)$ from CAT_n to $CAT_{n,k}$, and let $CAT\,\mathscr{H}_k(M)$ denote the set of homotopy classes of *CAT* k-fields on M.

Let R be a reduction of $\tau(M)$ from CAT_n to $CAT_{n,k}$. Then $\tau(M)$ contains a trivial k-dimensional *CAT* subbundle ϵ^k. Thus, there is a neighborhood V_1 of the diagonal $\Delta(M)$ in $M \times M$, and a *CAT* homeomorphism $h\colon V_1 \cap E(\epsilon^k) \to M \times R^k$ such that the following diagram commutes

$$\begin{tikzcd}
 & V_1 \cap E(\epsilon^k) \arrow[dr, "\pi_1"] \arrow[dd, "h"] & \\
M \arrow[ur, "\Delta"] \arrow[dr, "\mathrm{id}\times\mathbf{O}"'] & & M \\
 & M \times R^k \arrow[ur, "\mathrm{proj}"'] &
\end{tikzcd}$$

Define $F\colon M \times R^k \to M \times M$ by $F(b, r) = h^{-1}(b, r)$ and let $p\colon M \to \mathscr{E}_{CAT}(R^k, M)$ be given by $p(b)(r) = \pi_2 F(b, r)$. The assignment $R \mapsto p$ determines a well-defined function $\theta\colon CAT\,\mathscr{R}_k(M) \to CAT\,\mathscr{H}_k(M)$.

We show that θ is onto as the injectivity of θ is easily proven. Let $p_0, p_1\colon M \to \mathscr{E}_{CAT}(R^k, M)$ be homotopic *CAT* k-fields with $P\colon I \times M \to \mathscr{E}_{CAT}(R^k, M)$ the homotopy connecting them. By Theorem 1 of Cernavskii [5], if $n - k \neq 2$ and $n \geq 5$, for a sufficiently small compact neighborhood K_b of b in M there is a level preserving *TOP* homeomorphism $Q\colon I \times K_b \times M \to I \times K_b \times M$ such that $Q(t, x, P(t, x)(r)) = (t, x, P(t, b)(r))$ for all $x \in K_b$, $t \in I$, $r \in R^k$.

Cernavskii's covering isotopy theorem for *TOP* locally flat embeddings [5] implies that there exists a 1-cell $Q'\colon I \times M \to I \times M$ such that $(t, P(t, b)(r)) = Q'(t, P(0, b)(r))$ for all $t \in I$, $r \in R^k$.

If $CAT = PL$ and $n - k \geq 4$, Q and Q' can be replaced by a *PL* level preserving *PL* homeomorphism by using Miller's sliced approximation theorem [26]. As noted in the proof of Theorem 2.5, we can assume $n - k \geq 3$ by the variation of Miller's sliced approximation theorem where the domain is a manifold (see [40]).

Since $P(0, b)$ is a *CAT* locally flat embedding, there exists a neighborhood U of $\mathbf{O}$ in R^n such that $P(0, b)$ has a *CAT* extension $S\colon U \to M$. Define $\bar{P}\colon I \times K_b \times U \to I \times K_b \times M$ by

$$\bar{P}(t, x, r) = Q^{-1}(t, x, Q'(t, S(r))).$$

We have thus extended $P: I \times M \to \mathscr{E}_{CAT}(R^k, M)$ locally to a map $P^*: I \times K_b \to \mathscr{E}_{CAT}(U, M)$. A relative coring theorem for CAT microbundles [22] then yields a homotopy class of reductions of $\tau(M)$ from CAT_n to $CAT_{n,k}$.

§4. NORMAL BUNDLES

Let M^n be a locally flat CAT submanifold of a CAT manifold N^{n+q}. A *CAT normal R^q-bundle* of M in N is a pair (M_0, π_0) where M_0 is a neighborhood of M in N and $M \hookrightarrow M_0 \xrightarrow{\pi_0} M$ is a CAT R^n-bundle. Two such CAT normal bundles, say (M_0, π_0) and (M_1, π_1) are *CAT concordant* if there is a CAT normal bundle of $M \times I$ in $N \times I$, namely $M \times I \xrightarrow{\pi} M' \to M \times I$, such that $(\pi^{-1}(M \times \{i\}), \pi|M \times \{i\}) = (M_i, \pi_i)$ for $i = 0, 1$. If we further require that $(\pi^{-1}(M \times \{i\}, \pi|M \times \{i\})$ is a CAT normal bundle of $M \times \{i\}$ in $N \times \{i\}$ for all $i \in I$, then (M_0, π_0) and (M_1, π_1) are said to be *CAT isotopic.*

Let CAT^n_{n+q} denote the semi-simplicial group of CAT homeomorphisms of R^{n+q} onto itself leaving $\mathbf{O}$ fixed and $R^n \subset R^{n+q}$ invariant.

THEOREM 4.1. *Let $M^n \subset N^{n+q}$ be CAT manifolds, M CAT locally flat in N. There is a* 1–1 *correspondence between CAT normal bundles of M in N and homotopy classes of cross-sections of the bundle associated to $\tau(N)|M$ with fiber $CAT^n_{n+q}/CAT_q \times CAT_n$.*

For $CAT = PL$ Theorem 4.1 was proven by Haefliger and Wall [10]. The same proof in conjunction with the topological immersion theorem [9] establishes Theorem 4.1 when $CAT = TOP$.

By the PL and TOP isotopy extension theorem [6], $CAT_{n+q,n} \subset CAT^n_{n+q} \xrightarrow{r} CAT_n$ is a fibration, where r is restriction to R^n. The homotopy sequence of the square $(CAT^n_{n+q}; CAT_{n+q,n}, CAT_n \times CAT_q; CAT_q)$ then yield

LEMMA 4.2. $\pi_*(CAT^n_{n+q}, CAT_n \times CAT_q) \cong \pi_i(CAT_{n+q,n}, CAT_q)$.

THEOREM 4.3. *If $q \geq 3$, then $\pi_i(CAT_{n+q,n}, CAT_q) \cong \pi_i(CAT_{n+q}, O_{n+q}, CAT_q; O_q)$ for $i < 2q - 4$.*

Proof. Consider the diagram

$$\begin{array}{ccccc} CAT_{n+q} & \subset & CAT_{n+q,n} & \subset & CAT_q \\ \cup & \square_3 & \cup & \square_1 & \cup \\ O_{n+q} & \supset & O_q & \supset & O_q \end{array}$$

and the homotopy sequence

$$\to \pi_i(\square_1) \to \pi_i(\square_2) \to \pi_i(\square_3) \to$$

where $\square_2$ is the outside square. Theorem 2.5 implies that if $q \geq 3$, then $\pi_i(CAT_{n+q,n}, CAT_q) \cong \pi_i(\square_1) \cong \pi_i(\square_2)$ for $i < 2q - 4$.

As a consequence of the stability Theorems 0.1 and 0.2 we have

COROLLARY 4.4. *If $q \geq 3$, then $\pi_i(CAT_{n+q,n}, CAT_q) = 0$ for either*

(i) *$CAT = PL$ and $i \leq \min(q + 1, 2q - 5)$, or*

(ii) *$i \leq q + j$ and $q \geq 5 + j$, where $j = 0, 1, 2$.*

Together with Theorem 4.1 and Lemma 4.2 we have

THEOREM 4.5 *Let M^n be a CAT locally flat submanifold of a CAT manifold N^{n+q}. If either* (i) *$CAT = PL$ and $q \geq \max(n - 2, (n + 2)/2)$ or* (ii) *$q \geq n - j - 1$ and $q \geq 5 + j$ where $j = 0, 1, 2$, then M has a CAT normal bundle in N. If either* (a) *$CAT = PL$ and $q \geq \max(n - 1, (n + 4)/2)$ or* (b) *$q \geq n - j$ and $q \geq 5 + j$ where $j = 0, 1, 2$, then it is unique up to isotopy.*

Remark 4.6. Rourke and Sanderson have obtained this result for $q \geq n$ [33].

Remark 4.7. Theorem 4.5 when combined with Corollary 1.3 produces results concerning existence and uniqueness of normal disk bundles.

Remark 4.8. There are several examples of CAT locally flat submanifolds with no CAT normal bundles [12], [13]. Hence $\pi_i(CAT_{n,k}, CAT_{n-k}) \neq 0$ for some i, n, k. Thus, there exists bundles over a complex which contain a trivial bundle but do not split it off. This proves Proposition 2.2(b).

Lower codimensional obstructions for CAT normal bundles

Let $CATN(S^n, S^{n+q})$ denote the set of isotopy classes of CAT normal bundles of S^n naturally embedded in S^{n+q}. Theorem 4.1, Lemma 4.2, and obstruction theory then imply

PROPOSITION 4.9. *There is a natural one to one correspondence between the elements of $CATN(S^n, S^{n+q})$ and the elements of $\pi_n(CAT_{n+q,n}, CAT_q)$.*

If we let $\widetilde{CAT}N(S^n, S^{n+q})$ denote the set of concordance classes of CAT normal bundles of S^n in S^{n+q}, then [32], Theorem 4.9 of [33] and the fact that $\pi_*(\widetilde{CAT}_q, CAT_q)$ are the obstruction groups to making a CAT block bundle a CAT R^q-bundle establish

PROPOSITION 4.10. *There is a natural one–one correspondence between the elements of $\widetilde{CAT}N(S^n, S^{n+q})$ and the elements of $\pi_n(\widetilde{CAT}_q, CAT_q)$, where if $CAT = TOP$ we further require that $q \geq 5$.*

We now show that for $q \geq 9$ the uniqueness part of Theorem 4.5 is the best possible.

THEOREM 4.11. *If $q \geq 9$, then $CATN(S^{q+3}, S^{2q+3}) = \widetilde{CAT}N(S^{q+3}, S^{2q+3}) = Z_2 \oplus \mathrm{Wh}^3(0)$.*

Proof. Rourke and Sanderson have shown that $\pi_i(O_{n+q}, O_q) \to \pi_i(\widetilde{CAT}_{n+q}, \widetilde{CAT}_q)$ is an isomorphism for $i < 2q-3$ ([32], [33]). Hence Theorem 2.5 and the exact sequences associated with the square $(\widetilde{CAT}_{n+q}; CAT_{n+q}, \widetilde{CAT}_{n+q,n}; CAT_{n+q,n})$ imply that $\pi_i(\widetilde{CAT}_{n+q,n}, CAT_{n+q,n}) \cong \pi_i(\widetilde{CAT}_{n+q}, CAT_{n+q})$ for $i < 2q-4$. Note that Corollary 4.4 and Propositions 4.9 and 4.10 show that $\pi_i(\widetilde{CAT}_q, CAT_q) = 0$ for $i \leq q+2$. The exact sequence of the triple $(\widetilde{CAT}_{n+q,n}, CAT_{n+q,n}, CAT_q)$ then yields that $\pi_i(CAT_{n+q,n}, CAT_q) \cong \pi_i(\widetilde{CAT}_{n+q,n}, CAT_q) = \pi_i(\widetilde{CAT}_q, CAT_q)$ for $i < \min(n+q+2, 2q-5)$, so at least $\pi_{q+3}(CAT_{2q+3,q+3}, CAT_q) \cong \pi_{q+3}(\widetilde{CAT}_q, CAT_q)$ which establishes the equality $CATN(S^{q+3}, S^{2q+3}) = \widetilde{CAT}N(S^{q+3}, S^{2q+3})$, for $q \geq 9$. The fact that $CATN(S^{q+3}, S^{2q+3}) = Z_2 \oplus \mathrm{Wh}^3(O)$ follows from Theorems 0.3, 4.3 and 4.9.

Using these results and the results of §1, one can similarly prove results concerning concordance classes of normal disk bundles. In particular, if $\widetilde{CAT}N(I)(S^n, S^{n+q})$ denotes the set of concordance classes of CAT normal disk bundles of S^n in S^{n+q}, then one can establish that $\widetilde{CAT}N(I)(S^n, S^{n+q}) \cong \pi_n(\widetilde{CAT}_n, CAT_n(I)) \cong \pi_n(CAT_{n+q}; O_{n+q}, CAT_n(I); O_n) \cong \pi_n(CAT_{n+q}; O_{n+q}, CAT_{n-1}; O_{n-1})$ for $n < 2q-4$. Thus, Theorem 0.3 yields that if $q \geq 9$, then $CATN(I)(S^{q+2}, S^{2q+2}) = Z_2 \oplus \mathrm{Wh}^3(O)$, so that there exists a CAT normal disk bundle η_0 of S^{q+2} in S^{2q+2} which is not CAT concordant to the standard normal disk bundle η_1. If η_0 is a trivial bundle, then, using the standard construction of Hirsch [13] (which can also be adapted when $CAT = TOP$), there exists a CAT $(2q+3)$-manifold M and a CAT locally flat embedding of S^{q+3} in M with no CAT normal disk bundle (even though it has a CAT normal R^q-bundle). Here,

$$M = D^{q+3} \times D^q \underset{F_0}{\cup} S^{2q+2} \times I \underset{F_1}{\cup} D^{q+3} \times D^q \text{ where } F_i\colon S^{q+2} \times D^q \to E(N_i) \subset S^{2q+2} \times \{i\},\ i = 0, 1,$$

are the trivilization of the non-concordant normal bundles N_0 and N_1, and the cores of the bundles $D^{q+3} \times D^q$ union $S^{q+2} \times I$ form the resulting embedded S^{q+3}.

To show that such a CAT normal disk bundle N_0 of S^{q+2} in S^{2q+2} exists is equivalent to showing that the boundary homomorphism $\partial\colon \pi_{q+2}(\widetilde{CAT}_q, CAT_q(I)) \to \pi_{q+1}(CAT_q(I))$ of the homotopy sequence of the pair $(\widetilde{CAT}_q, CAT_q(I))$ has non-trivial kernel. Via diagram chasing, one can establish that this is indeed the case if $\pi_{q+2}(CAT_{q-1}, O_{q-1}) \to \pi_{q+2}(CAT_q, O_q)$ is not onto. Thus,

PROPOSITION 4.13. *Suppose $q \geq 9$. If the homomorphism $\pi_{q+2}(CAT_{q-1}, O_{q-1}) \to \pi_{q+2}(CAT_q, O_q)$ induced by the stability map is not onto, then there exists a CAT $(2q+3)$-manifold M and a CAT locally flat embedding of S^{q+3} in M with a CAT normal R^q-bundle, but no CAT normal disk bundle.*

To construct a CAT $(2q+4)$-manifold and a CAT locally flat embedding of S^{q+4} in M with no CAT normal R^q-bundle seems to be a much harder problem.

§5. INTEGRATING CAT VECTOR FIELDS

Let M be a closed CAT n-manifold. A *codimension q CAT foliation* (also called a $(n-q)$-foliation) on M consists of an open covering $\{U_i\}_{i \in J}$ of M and a family of CAT submersions $\{f_i\colon U_i \to R^q | i \in J\}$ such that for every $x \in U_i \cap U_j$ there exists a CAT homeomorphism g_{ji} mapping a neighborhood of $f_i(x)$ onto a neighborhood of $f_j(x)$ such that $f_j = g_{ji} f_i$ in a neighborhood of x.

Since each f_i is a CAT submersion, we have that for each $x \in U_i$, $f_i^{-1}(f_i(x))$ is a codimension q CAT submanifold of U_i. The *leaf topology* on U_i comes from considering U_i as the disjoint

union of the codimension q CAT submanifolds $\{f_i = \text{constant}\}$. The overlap condition implies that the leaf topologies coincide on overlapping members of the open cover $\{U_i\}_{i \in J}$ so that they yield a topology, called the *leaf topology*, on M. A connected component of M under this topology is called a *leaf* of the foliation.

If M is a $DIFF$ manifold with a non-zero differentiable vector field s, then we can integrate s to yield a $DIFF$ 1-foliation $\mathcal{F}$ on M such that for all $x \in M$, $s(x)$ lies in the tangent bundle of the leaf of $\mathcal{F}$ through x.

Definition 5.1. Let M be a CAT manifold. A non-zero CAT vector field $s: M \to \tau(M) \subset M \times M$ is said to be *integrable* if there exists a CAT 1-foliation $\mathcal{F}$ on M such that for all $b \in M$, $\pi_2 s(b)$ lies on the leaf of $\mathcal{F}$ which passes through b.

It is asking too much that all non-zero CAT vector fields be integrable, for let M be a $DIFF$ manifold. If s is a C^0-vector field on M, then s determines a non-zero TOP vector field on M which is not always integrable, as the solutions to the associated differential equations need not yield a 1-foliation on M.

In this section we prove that a non-zero CAT vector field on a closed CAT manifold is homotopic to an integrable CAT vector field.

THEOREM 5.2. *Let M be a closed CAT n-manifold. If $CAT = PL$ or $CAT = TOP$ and $n \neq 4, 5$, then any non-zero CAT vector field on M is homotopic to an integrable one.*

Proof. Let $s: M \to \tau(M)$ be a non-zero CAT vector field on M. By Corollary 1.5, $\tau(M)$ is CAT bundle isomorphic to $\xi^{n-1} \oplus \epsilon^1$. Let $U, V \subset M$ be coordinate charts with $U \subset V$. Then Gauld's submersion theorem [9] gives a CAT submersion $p: M - clU \to R^1$ which determines a CAT codimension one foliation $\mathcal{G}_0$ on $M - clU$. Now V has an induced $DIFF$ structure Σ_0 on it, so that by the proof of Theorem 3.1 of [19] there is an ambient isotopy $h_t: M - clU \to M - clU$ such that $h_1(\mathcal{G}_0)|V - clU$ is a $DIFF$ (with respect to Σ_0) codimension one foliation on the annulus $V - clU$. Let $\mathcal{F}_0$ be a $DIFF$ 1-foliation on $V - clU$ transverse to $\mathcal{G}_0$ (just take a normal line field and integrate). By Siebenmann's theorem 6.25 of [35] there exists a CAT 1-foliation $\mathcal{F}$ transverse to $h_1(\mathcal{G}_0)$ such that $\mathcal{F}|V - clU = \mathcal{G}_0$. Let $s_0: M - clU \to \tau(M - clU)$ be the non-zero CAT vector field on $M - clU$ determined by $\mathcal{F}$. Note that s_0 is homotopic to $s|M - clU$ and thus extends to a non-zero CAT vector field $s_1: M \to \tau(M)$ which is also homotopic to s. Approximate $s_1|V$ by a $DIFF$ (with respect to Σ_0) vector field rel $V - clU$ and integrate the resulting vector field to yield a non-zero CAT vector field $s_2: M \to \tau(M)$ homotopic to s and *a* CAT 1-foliation $\mathcal{F}_2$ on M such that for all $b \in M$, $\pi_2 s_2(b)$ lies on the leaf of $\mathcal{F}_2$ through b.

Remark. If M is a $DIFF$ or PL manifold and s is a non-zero TOP vector field, Theorem 5.2 follows from the results of §1 and the recent work of Thurston [42] on converting $B\Gamma_q^0$ structures ($q > 1$) to codimension q foliations.

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