



Families of simply connected 4-manifolds with the same Seiberg–Witten invariants

Ronald Fintushel^{a,*}, Ronald J. Stern^{b,2}

^a*Department of Mathematics, Michigan State University, East Lansing, MI 48824, USA*

^b*Department of Mathematics, University of California, Irvine, CA 92697, USA*

Received 4 January 2003; accepted 1 March 2004

Abstract

This article presents several new constructions of infinite families of smooth 4-manifolds with the property that any two manifolds in the same family are homeomorphic. While the construction gives strong evidence that any two of these manifolds are not diffeomorphic, they cannot be distinguished by Seiberg–Witten invariants. Whether these manifolds are, or are not, diffeomorphic seems to be a very difficult question to answer.

For one of these constructions, each member of the family is symplectic with the further property that each contains nullhomologous tori with the property that infinitely many log transformations on these tori yield nonsymplectic 4-manifolds. This is detected by calculations of Seiberg–Witten invariants. The surgery in question can be performed on any 4-manifold which contains as a codimension 0 submanifold a punctured surface bundle over a punctured surface and a nontrivial loop in the base which has trivial monodromy.

A starting point for another class of examples in this paper is a family of examples which show that the Parshin–Arakelov theorem for holomorphic Lefschetz fibrations is false in the symplectic category. Such families are constructed by means of knot surgery on elliptic surfaces. It is shown that for a fixed homeomorphism type X (of a simply connected elliptic surface) and a fixed integer $g \geq 3$, there are infinitely many genus g Lefschetz fibrations on nondiffeomorphic 4-manifolds, all homeomorphic to X .

© 2004 Elsevier Ltd. All rights reserved.

Keywords: 4-Manifold; Seiberg–Witten invariant

* Corresponding author. Tel.: +1-517-353-8146.

E-mail address: ronfint@math.msu.edu (R. Fintushel).

¹ Partially supported by NSF Grant DMS0072212.

² Partially supported by NSF Grant DMS9971667.

1. Introduction

In 1994, the introduction of the Seiberg–Witten equations quickly fostered optimism in 4-manifold topology. Invariants derived from these equations immediately led to the solutions of several outstanding conjectures, and it was felt that the classification of 4-manifolds was finally in sight. After 8 years, however, the opposite seems to be true. Topological constructions, along with the Seiberg–Witten invariants, have demonstrated that simply connected smooth 4-manifolds are much more complicated than earlier envisioned, and the classification of smooth 4-manifolds has retreated beyond the visible horizon. In this paper, we add to this quagmire and exhibit constructions which yield new infinite families of homeomorphic simply connected 4-manifolds, all of which have the same Seiberg–Witten invariants. These manifolds have many characteristics which lead to the belief that they ought not be diffeomorphic, but this remains unsettled. We also introduce a construction which readily converts an irreducible symplectic 4-manifold X to a family of infinitely many irreducible nondiffeomorphic but mutually homeomorphic smooth 4-manifolds. Exactly one member of this family will admit a symplectic structure.

There are many examples of the first sort which are already known. For example, for each odd integer $r \geq 3$, one can consider the Horikawa surfaces with holomorphic euler number $2r - 1$ and $c_1^2 = 4r - 8$ (see e.g. [12]). For each such r , there are two deformation classes of these simply connected complex algebraic surfaces, and the two deformation types are homeomorphic. Since the Horikawa surfaces are general type, their Seiberg–Witten invariants are $\text{SW} = t_K - t_K^{-1}$ where K is the canonical class (which is primitive). (We shall view the Seiberg–Witten invariants of simply connected 4-manifolds as elements of the integral group ring $\mathbb{Z}H_2(X; \mathbb{Z})$. The notation t_b will be used to denote the element of the group ring corresponding to $b \in H_2(X; \mathbb{Z})$ and its (integer) coefficient will at times be denoted by $\text{SW}_X(b)$.) Thus, the Seiberg–Witten invariants fail to distinguish these manifolds. Whether or not they are indeed diffeomorphic is an extremely interesting open question.

Knot surgery [7,8] can be used to produce infinite families of homeomorphic simply connected 4-manifolds with the same Seiberg–Witten invariants. For example, if we produce a family of knots $\{K_n\}$ all of which have the same symmetrized Alexander polynomial $\Delta_{K_n}(t)$ (for example, take an infinite list of knots of Alexander polynomial equal to 1) then the manifolds X_n obtained from knot surgery on an elliptic fiber of the K3-surface will all be homeomorphic to K3 and will have the same Seiberg–Witten invariants. Although these examples seem likely to be nondiffeomorphic, one needs to use care here. Akbulut [1] has shown that for any knot K that the knot surgery manifolds arising from K3 by using K and its mirror image $-K$ are diffeomorphic. The proof uses special symmetry properties, but it indicates that one should not assume too quickly that the knot surgery manifolds arising from different knots ought to be nondiffeomorphic.

In this paper, we shall produce three further families. The first is a generalization of the authors' construction [9] of nonsymplectic manifolds with one basic class. For this generalization, we begin by constructing families of homeomorphic simply connected symplectic manifolds with one basic class with Seiberg–Witten invariant ± 1 , i.e. with $\text{SW} = t_K \pm t_K^{-1}$. As evidence that these manifolds with one basic class are never diffeomorphic to a complex surface, we show that this construction yields both families of manifolds homeomorphic to a simply connected complex surface as well as families of manifolds that are homeomorphic to no complex surface.

We then twist this construction to yield further families whose Seiberg–Witten invariants have similar properties. In addition, in each manifold of this family, we find nullhomologous tori with

the property that a $(-1/m)$ -logarithmic transform on any of them multiplies the Seiberg–Witten invariant by $m + 1$. This results in families of homeomorphic simply connected 4-manifolds with $SW = (m + 1)(t_K \pm t_K^{-1})$. We conjecture that no two of these manifolds are diffeomorphic. Note that if $m \neq 0$ these manifolds are nonsymplectic. Thus, this construction shows how one can perform surgery to make a symplectic manifold nonsymplectic (cf. [20]).

The third collection of families builds on a construction of the authors which gives a counterexample to the existence of a symplectic Parshin–Arakelov theorem. In particular, we show that for a fixed even (resp. odd) integer g there are infinitely many distinct genus g Lefschetz fibrations on a homotopy rationally elliptic (resp. K3) surface. These examples have been promised for some time and were first presented in a talk of the first author at the 1998 Aarhus Topology Conference.

2. Construction I

Fix a simply connected symplectic 4-manifold X , and let C be a smoothly embedded symplectic surface in X which has genus $n \geq 2$ and self-intersection 0. Given X , a positive integer g , and a particular genus g fiber bundle Y over a genus n surface constructed below, we shall associate a symplectic simply connected 4-manifold $Z = Z(X, C, g)$ with $c_1^2(Z) = c_1^2(X) + 8g(n - 1)$, $\chi(Z) = \chi(X) + g(n - 1)$, and with $SW = t_K - t_K^{-1}$. Here $\chi = \chi(X)$ is one-fourth the sum of the Euler characteristic and the signature of X , which is the holomorphic Euler characteristic in the case that X is a complex surface. We shall choose a family of manifolds X_j , homeomorphic to X , and distinguished by their Seiberg–Witten invariants. Each X_j contains a genus n symplectic surface C_j with self-intersection 0. We fiber sum along these surfaces to obtain the family $Z(X_j, C_j, g)$ with the same Seiberg–Witten invariants.

To construct the bundle Y , let K_g denote the $(2g + 1, -2)$ -torus knot, pictured in Fig. 1, and let M_{K_g} denote the 3-manifold obtained by performing 0-framed surgery on K_g . This manifold has the integral homology of $S^2 \times S^1$. In Fig. 1 we can see an obvious genus g Seifert surface for K_g . In M_{K_g} , fix a closed genus g surface Σ_g obtained from capping off this Seifert surface with a disk.

Now fix a $g \geq 1$, and set $K = K_g$ and $\Sigma = \Sigma_g$. Because K is a fibered knot, $S^1 \times M_K$ is a symplectic manifold which is fibered over T^2 with symplectic fiber Σ . The fiber sum

$$Y_{2,g} = S^1 \times M_K \#_{\Sigma} S^1 \times M_K$$

of two copies of $S^1 \times M_K$ is again a symplectic manifold. This manifold has euler number $e(Y_{2,g}) = 4g - 4$ and signature $\text{sign}(Y_{2,g}) = 0$. In the j th copy of $S^1 \times M_K$, let T_j denote the torus $S^1 \times m$ where m is a meridian to K (and thus generates $H_1(M_K; \mathbb{Z})$). This torus is a section to the fibration of $S^1 \times M_K$. It follows from work of Meng and Taubes [14] that any basic class of $S^1 \times M_K$ must be a multiple of T_j . They show that the Seiberg–Witten invariant of $S^1 \times M_K$ is $\Delta_K(t^2)/(t - t^{-1})^2$ where $t = t_{T_j} \in \mathbb{Z}H_2(S^1 \times M_K; \mathbb{Z})$ and Δ_K is the symmetrized Alexander polynomial of K . Note that the intersection number of T_j and Σ is $T_j \cdot \Sigma = 1$.

We wish to determine the Seiberg–Witten invariant of $Y_{2,g}$. Write

$$Y_{2,g} = ((S^1 \times M_K) \setminus (\Sigma \times D^2)) \cup ((S^1 \times M_K) \setminus (\Sigma \times D^2)).$$

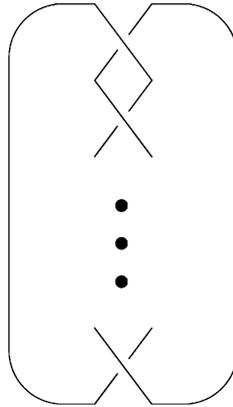


Fig. 1.

In $S^1 \times M_K$, the torus T_j intersects $\Sigma \times D^2$ in some $\{y_j\} \times D^2$. We may assume that the identification of the two copies of Σ in the fiber sum is chosen so that y_1 is identified with y_2 . We then obtain a genus 2 surface $(T_1 \setminus D^2) \cup (T_2 \setminus D^2)$ in the fiber sum. (Of course, $Y_{2,g}$ is a fiber bundle over a genus 2 surface, and we have just constructed a section.) Let $\tau_{1,2}$ denote the class of this surface in $H_2(Y_{2,g}; \mathbb{Z})$. Notice that $\tau_{1,2} \cdot \Sigma = 1$ (and that $\tau_{1,2}^2 = 0 = \Sigma^2$).

There are also $4g$ classes in $H_2(Y_{2,g}; \mathbb{Z})$ which are obtained as follows: for a fiber $\Sigma \times \{t\} \subset \partial(\Sigma \times D^2)$, the inclusion $H_1(\Sigma \times \{t}; \mathbb{Z}) \rightarrow H_1((S^1 \times M_K) \setminus (\Sigma \times D^2); \mathbb{Z})$ is trivial. Thus, the identification of fibers and a collection of loops $\{a_i\}$ on Σ which gives a basis for $H_1(\Sigma; \mathbb{Z})$ gives rise to $2g$ two-dimensional homology classes V_i formed from unions of bounding surfaces. We shall refer to these as *vanishing classes*. For each a_i there is a rim torus $R_i = a_i \times \partial D^2$. With appropriate orientation choices, we have for all i, j :

$$\begin{aligned} R_j \cdot V_i &= a_i \cdot a_j \quad (\text{the skew-symmetric intersection form on } H_1(\Sigma; \mathbb{Z})), \\ R_j \cdot \Sigma &= 0, \\ R_j \cdot \tau_{1,2} &= 0. \end{aligned}$$

The intersection matrix Q of $\{V_i, R_j | i, j = 1, \dots, 2g\}$ has determinant 1, and these classes along with $\tau_{1,2}$ and Σ form a basis for $H_2(Y_{2,g}; \mathbb{Z})$.

It will be useful to have a more specific description of the vanishing classes V_j . As we have mentioned, the complement in S^3 of the $(2g + 1, -2)$ -torus knot K_g , fibers over the circle, and its fiber is a punctured genus g surface, Σ'_g . If we make the obvious choices for the loops $\{a_j\}$ which are mentioned above, their associated Seifert linking pairing is

$$\ell k(a_i, a_j^+) = \begin{cases} 1, & j = i, i + 1, \\ 0, & \text{otherwise,} \end{cases}$$

where a_j^+ is the positive push-off of a_j . Notice that a_j^+ bounds a genus 1 surface in $S^3 \setminus K_g$. A typical element a_1 of this basis and its push-off a_1^+ for K_1 , the left-hand trefoil knot, are shown in Figs. 2 and 3.

Again writing K for K_g , we consider the fiber bundle

$$p : (S^1 \times M_K) \setminus (\Sigma \times D^2) \rightarrow T^2 \setminus D^2.$$

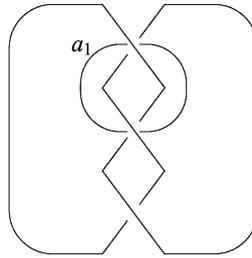


Fig. 2.

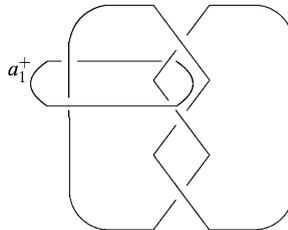


Fig. 3.

The base deformation retracts to a wedge of two circles, $S_1^1 \vee S_2^1$, where $p^{-1}(S_1^1) = S_1^1 \times \Sigma$ and $p^{-1}(S_2^1) = M_K$. Let $S_1^1 \cap S_2^1 = \{x_0\}$, and let y_0 be a point in the boundary of $T^2 \setminus D^2$. Let $\delta : [0, 1] \rightarrow T^2 \setminus D^2$ parametrize S_2^1 so that $\delta(0) = x_0$ and so that the positive push-off a_i^+ lies in $p^{-1}(\delta(1))$.

Now let γ be an embedded path in $T^2 \setminus D^2$ with $\gamma(0) = y_0$ and $\gamma(1) = \delta(1)$. In $p^{-1}(y_0) \cong \Sigma$ consider the circle $a_{i,\delta}$, the obvious circle in the fiber over y_0 . This is a boundary component of an embedded annulus in $p^{-1}(\gamma)$ whose other boundary component is a_i^+ , which bounds a genus 1 surface in $M_K = p^{-1}(S_2^1)$. We thus obtain in $(S^1 \times M_K) \setminus (\Sigma \times D^2)$ a genus 1 surface $G_{i,1}$ with boundary a_i . Similarly, in the second copy of $(S^1 \times M_K) \setminus (\Sigma \times D^2)$, we have a similar genus 1 surface $G_{i,2}$.

So far, we have left ourselves the freedom of choosing the gluing map which constructs the fiber sum. At this point, we specify that the chosen gluing must take the boundary of $G_{i,1}$ diffeomorphically onto the boundary of $G_{i,2}$ for each i . Then $V_i = G_{i,1} \cup G_{i,2}$ is a genus 2 surface in $Y_{2,g}$.

Lemma 1. *The genus 2 surface V_i has self-intersection number 2 and satisfies $V_i \cdot \Sigma = 0$, $V_i \cdot \tau_{1,2} = 0$, and $V_i \cdot R_j = a_i \cdot a_j$.*

Proof. Except for the calculation of the self-intersection number, everything is clear from the construction of V_i . Let $S_2^{1'}$ denote a circle in $T^2 \setminus D^2$ which is disjoint from and parallel to S_2^1 , and let γ' be a path parallel to γ from a point in $\partial(T^2 \setminus D^2)$ to a point on $S_2^{1'}$. Using the structure of the fibration, $G_{i,1}$ in $p^{-1}(\gamma \cup S_2^1)$ can be pushed onto $G'_{i,1}$ in $p^{-1}(\gamma' \cup S_2^{1'})$. Depending on the choices made for $S_2^{1'}$ and γ' , either S_2^1 will intersect γ' in a single point or $S_2^{1'}$ will intersect γ in a single positive point. In either case, the only fiber in which $G_{i,1}$ and $G'_{i,1}$ intersect is the fiber over this point. The intersection of each surface with the fiber is a push-off of a_i , and these two push-offs intersect once on this fiber. The same is true for surfaces $G_{i,2}$ and $G'_{i,2}$, implying that $V_i^2 = 2$. \square

The adjunction inequality implies that a basic class must intersect a smoothly embedded surface of genus h and self-intersection $2h - 2$ with intersection number equal to 0. In particular, the intersection number of any basic class of $Y_{2,g}$ with a rim torus must be 0. Consider a basic class k of $Y_{2,g}$ and write

$$k = t\tau_{1,2} + s\Sigma + \sum_{i=1}^{2g} u_i R_i + v_i V_i.$$

The intersections $k \cdot R_j = 0, j = 1, \dots, 2g$, give rise to the equation $Q^T \mathbf{v} = \mathbf{0}$ where Q^T is the transpose of the intersection matrix Q and $\mathbf{v} = (v_1, \dots, v_{2g})$. Since Q is nonsingular, all $v_i = 0$.

We see that a basic class of $Y_{2,g}$ has the form

$$k = t\tau_{1,2} + s\Sigma + \sum_{j=1}^{2g} u_j R_j.$$

Another application of the adjunction inequality gives $0 = V_i \cdot k$, and we get the equation $Q\mathbf{u} = 0$. Once again this means that each coefficient $u_j = 0$. Thus, each basic class of $Y_{2,g}$ has the form $k = t\tau_{1,2} + s\Sigma$.

Since $Y_{2,g}$ is a symplectic manifold, it has simple type; that is, for each basic class k ,

$$k^2 = 3 \operatorname{sign}(Y_{2,g}) + 2e(Y_{2,g}) = 8g - 8.$$

So if $k = t\tau_{1,2} + s\Sigma$ then $2st = 8g - 8$. Applying the adjunction inequality to $\tau_{1,2}$ and to Σ yields

$$2 \geq \tau_{1,2}^2 + |k \cdot \tau_{1,2}| = |s| \quad \text{and} \quad 2g - 2 \geq \Sigma^2 + |k \cdot \Sigma| = |t|.$$

Thus, if $g > 1$, the only basic classes of $Y_{2,g}$ are $\pm\beta$ where $\beta = (2g - 2)\tau_{1,2} + 2\Sigma$. Clearly, β is the canonical class of $Y_{2,g}$, and

$$\operatorname{SW}_{Y_{2,g}} = t_\beta + (-1)^{g-1} t_\beta^{-1}.$$

In the special case $g = 1$, $Y_{2,1}$ is a torus bundle over a genus 2 surface. In this case, the above argument only shows that for the canonical class $\beta = 2\Sigma$ of $Y_{2,1}$,

$$\operatorname{SW}_{Y_{2,1}} = t_\beta + c + t_\beta^{-1},$$

where $c = \operatorname{SW}_{Y_{2,1}}(0)$.

In general, let $Y_{n,g} = (S^1 \times M_K) \#_\Sigma (S^1 \times M_K) \#_\Sigma \cdots \#_\Sigma (S^1 \times M_K)$ (n copies) where $K = K_g$. Then $Y_{n,g}$ is a symplectic manifold which is a fiber bundle with a genus g fiber and a genus n base. It contains a section S of square 0 obtained from gluing together the punctured copies of $T_i = S^1 \times m_i$. We still denote the genus g fiber by Σ . Thus $S \cdot \Sigma = 1$. The same analysis as above shows that for $g > 1$ the only basic classes of $Y_{n,g}$ are $\pm\beta$ where $\beta = (2g - 2)S + (2n - 2)\Sigma$ is the canonical class of $Y_{n,g}$, and

$$\operatorname{SW}_{Y_{n,g}} = t_\beta + (-1)^{(g-1)(n-1)} t_\beta^{-1}.$$

Also $\operatorname{SW}_{Y_{n,1}}$ is a monic symmetric Laurent polynomial in powers of t_Σ , and the highest power that occurs is $\pm(2n - 2)$. These terms correspond to $\pm\beta$ where $\beta = (2n - 2)\Sigma$ is the canonical class.

We note for use below that when $n > 2$, we have $Y_{n,g} = \hat{Y}_1 \cup \cdots \cup \hat{Y}_n$, where each $\hat{Y}_i, i \neq 1, n$, is a bundle over $T^2 \setminus (D^2 \cup D^2)$. In these latter manifolds there are punctured tori $G_{i,j}^\pm$ so that we get genus 2, self-intersection 2 (vanishing) surfaces $V_{i,j} = G_{i,j}^+ \cup G_{i,j+1}^-$, for $i = 1, \dots, 2g$ and $j = 1, \dots, n - 1$.

Recall that the data given for our construction consists of a simply connected symplectic 4-manifold X with an embedded symplectic surface C of genus $n \geq 2$ and self-intersection number 0, and the integer $g \geq 1$. Form the symplectic manifold $Z(X, C, g)$ as the fiber sum of X and $Y_{n,g}$

$$Z = Z(X, C, g) = X \#_{C=S} Y_{n,g}.$$

Proposition 2. *If $\pi_1(X \setminus C) = 1$, then $\pi_1(Z) = 1$.*

Proof. The i th $\pi_1(S^1 \times M_K)$ is normally generated by the image of $\pi_1(T_i)$. Since T_i intersects Σ in a single point, a normal circle to Σ lies in T_i . Thus, $\pi_1(S^1 \times M_K \setminus \Sigma)$ is normally generated by the image of $\pi_1(T_i \setminus \text{point})$. An inductive application of Van Kampen’s theorem shows that $\pi_1(Y_{n,g})$ is normally generated by the image of $\pi_1(S)$. Thus, if $\pi_1(X \setminus C) = 1$, we have $\pi_1(Z) = \pi_1(Y_{n,g} \setminus S) / \pi_1(S \times S^1) = 1$. \square

Lemma 3. *Let S be an orientable surface in the compact orientable 4-manifold Y . The subgroup of $H_2(Y \setminus S; \mathbb{Z})$ generated by the rim tori of S is isomorphic to $H^1(S) / \text{im}(H^1(Y))$. It follows that the rim tori of $Z(X, C, g)$ arising from loops in $C = S$ are all nullhomologous in $Z(X, C, g)$.*

Proof. The group generated by the classes of rim tori is the kernel of $H_2(Y \setminus S) \rightarrow H_2(Y)$. Let N be a tubular neighborhood of S . Then the main statement of the lemma follows from

$$H_3(Y) \rightarrow H_3(Y, Y \setminus S) \xrightarrow{\partial} H_2(Y \setminus S) \rightarrow H_2(Y),$$

once we note that $H_3(Y, Y \setminus S) \cong H_3(N, \partial N)$ and invoke Poincaré duality. The second statement follows since $H^1(Y_{n,g}) \cong H^1(S)$. \square

Dually, in the situation of the lemma, there is a homomorphism $H_3(Y) \rightarrow H_1(S)$ given by intersection. The lemma states that the group generated by the rim tori is the quotient of $H_1(S)$ by the image of $H_3(Y)$. Note that in the special case where $p : Y \rightarrow B$ is a surface bundle over a surface, and S is the image of a section, this implies that the subgroup of $H_2(Y \setminus S)$ generated by the rim tori of S vanishes.

For a basic class κ of X , the adjunction inequality implies that the maximal intersection of κ with C is $\kappa \cdot C = 2n - 2$. This intersection number is achieved when κ is the canonical class of Z , and whenever this maximal intersection number is achieved, there is a surface B_κ in X , representing κ , which intersects C positively in exactly $2n - 2$ points. We can then form the class $\beta_\kappa \in H_2(Z, \mathbb{Z})$ which is represented by the union of B_κ with $2n - 2$ disks removed and a smooth surface representing β with the $2n - 2$ normal disks removed at the points where it intersects S . Any basic class α of Z satisfies

$$\alpha^2 = c_1^2(Z) = c_1^2(X) + c_1^2(Y_{n,g}) + 8n - 8.$$

Note that $\varepsilon_\kappa = \beta_\kappa + 2C$ has exactly this square.

It follows from [16] (and the fact that there are no rim tori) that

$$\text{SW}_Z(\varepsilon_\kappa) = \text{SW}_{Y_{n,g}}(\beta) \cdot \text{SW}_X(\kappa) = \text{SW}_X(\kappa)$$

and that these are the only basic classes of Z which have intersection number $2n - 2$ with C . To express this, write the Seiberg–Witten invariant of Z as

$$\text{SW}_Z = \text{SW}_{\{Z; C, -\}} + \text{SW}_{\{Z; C, \max\}},$$

where the first summand consists of terms of the form bt_ℓ where $|\ell \cdot C| < 2n - 2$, and the latter summand consists of terms corresponding to basic classes whose intersection number with C is $\pm(2n - 2)$. Similarly, write

$$SW_X = SW_{\{X;C,-\}} + SW_{\{X;C,\max\}}.$$

Then we have:

Proposition 4. *Let X be a symplectic 4-manifold containing an embedded symplectic surface C of genus n and self-intersection 0. Suppose also that $\pi_1(X \setminus C) = 0$. Then for each $g \geq 1$, $Z = Z(X, C, g)$ is simply connected, and if the Seiberg–Witten invariant of X is*

$$SW_X = SW_{\{X;C,-\}} + \sum_{k \cdot C = 2n-2} c_k(t_k + (-1)^{\chi(X)} t_k^{-1}),$$

then the Seiberg–Witten invariant of Z is

$$SW_Z = SW_{\{Z;C,-\}} + \sum_{k \cdot C = 2n-2} c_k(t_{\varepsilon_k} + (-1)^{\chi(Z)} t_{\varepsilon_k}^{-1}).$$

Under special hypotheses on the manifold X , one can verify that $SW_{\{Z;C,-\}} = 0$. For example, let X be the knot surgery manifold obtained from the K3-surface by replacing a neighborhood of a torus fiber F with the manifold $S^1 \times (S^3 \setminus K')$, where K' is a fibered knot of genus $n - 1$. In the K3-surface let C' be a symplectically embedded torus homologous to a fiber plus a section. Then in X , a disk in C' is replaced with a Seifert surface for K' , and we get a symplectic surface C of genus n and self-intersection 0. In [9] it is shown that $X \setminus C$ is simply connected, and as in that paper, adjunction inequality arguments can be used to show that $SW_{\{Z;C,-\}} = 0$, and $SW_Z = t_F^{n-1} + t_F^{1-n}$. These constructions give the examples promised at the beginning of this section.

3. Other interesting $Z(X, C, g)$

In this section, we shall choose X to be a complex surface with C a holomorphically embedded submanifold. For a finite number of g we will show that $Z(X, C, g)$ has $SW = t_K - t_K^{-1}$, and by a theorem of Persson et al. [18], its homeomorphism type does not support any complex structure. For the remaining g , there are indeed complex manifolds homeomorphic to $Z(X, C, g)$. We conjecture that none of the $Z(X, C, g)$ have a compatible complex structure. The relevant result is the following somewhat surprising restriction on the geography of complex surfaces which support a spin structure, i.e. with vanishing second Stiefel–Whitney class.

Theorem 5 (Persson et al. [18]). *A simply connected spin surface with characteristic numbers*

$$2\chi \leq c_1^2 < 3(\chi - 5)$$

satisfies $c_1^2 = 2(\chi - 3)$ with $c_1^2 = 8k$ and k odd or $c_1^2 = \frac{8}{3}(\chi - 4)$ with $\chi \equiv 1 \pmod{3}$.

Let $H(m)$ be the spin Horikawa surface with $\chi = 8m - 1$ (and $c_1^2(H(m)) = 2\chi - 6$). It is well known that each $H(m)$ supports a genus 2 Lefschetz fibration. Let C be a fiber. Then the manifolds

$Z(m, g) = Z(H(m), C, g)$ are spin and

$$c_1^2(Z(m, g)) = 16m + 8g - 8,$$

$$\chi(Z(m, g)) = 8m + g - 1.$$

Thus, whenever we fix m and choose g so that the characteristic numbers of $Z(m, g)$ are restricted by Theorem 5, we obtain symplectic manifolds that support no complex structure. For the remaining g , there are complex manifolds in the homeomorphism type of $Z(m, g)$; however, we conjecture that no $Z(m, g)$ supports a complex structure.

One can verify that $X(m, g)$ supports no complex structure whenever $g < 8/5m - 2$ and if $m = 1 \pmod{3}$ then $g \not\equiv 0 \pmod{3}$, if $m = 2 \pmod{3}$ then $g \not\equiv 1 \pmod{3}$, or if $m = 0 \pmod{3}$ then $g \not\equiv 2 \pmod{3}$.

For example, there are no restricted examples starting with $H(1)$ and $H(2)$, i.e. $Z(1, g)$ and $Z(2, g)$ are homeomorphic to complex surfaces. However, $Z(3, g)$ is restricted when $g=1$, $Z(4, g)$ is restricted when $g = 1, 2, 4$, $Z(5, g)$ is restricted when $g = 2, 3$, $Z(6, g)$ is restricted when $g = 1, 3, 4, 6, 7$, etc.

Note that since $H(m)$ is the m -fold fiber sum of $H(1)$ along a genus 2 fiber, $Z(m, g)$ is the fiber sum of $Z(1, g)$ with $m - 1$ copies of $H(1)$.

4. Construction II

This construction is similar to the last, but has properties which will be useful when performing the surgeries described in the next section. Again, let us begin with a simply connected symplectic 4-manifold X containing an embedded symplectic surface C of genus $n \geq 2$ and self-intersection 0. For any $g \geq 1$ we can form the manifold $Z(X, C, g)$ as above. Suppose that we tweak this construction: as a warm-up, we begin with a simple case. Let $K = K_g$ and form a twisted fiber sum as follows:

$$Y'_{1,g,n-1} = (S^1 \times M_K) \#_{\Sigma_g=S_g} Y_{g,n-1} = Y_{1,g} \#_{\Sigma_g=S_g} Y_{g,n-1}.$$

The genus g section S_g of $Y_{g,n-1}$ is identified with the genus g fiber of $S^1 \times M_K$ in the fiber sum. This new symplectic manifold contains the genus n surface S' of self-intersection 0 obtained from the sum of the genus $n - 1$ fiber of $Y_{g,n-1}$ together with the genus 1 section S_1 of $Y_{1,g}$. Define $Z'_{1,g,n-1}(X, C) = X \#_{C=S'} Y'_{1,g,n-1}$.

Proposition 6. *If $X \setminus C$ is simply connected, then so is $Z'_{1,g,n-1}(X, C)$.*

Proof. As in the proof of Proposition 2, $\pi_1(Y_{1,g} \setminus \Sigma_g)$ is normally generated by the image of $\pi_1(S_1 \setminus \text{pt})$. Also, $\pi_1(Y_{g,n-1} \setminus S_g)$ is normally generated by the image of the fundamental group of a section S'_g , disjoint from S_g , and by the normal circle to S_g (which lies on a fiber). In $\pi_1(Y'_{1,g,n-1})$, the image of $\pi_1(S'_g)$ is identified with $\pi_1(\Sigma_g)$ in $\pi_1(Y_{1,g} \setminus \Sigma_g)$; thus, $\pi_1(Y'_{1,g,n-1})$ is normally generated by the image of $\pi_1(S')$. As in Proposition 2, if $\pi_1(X \setminus C)$ vanishes, it follows that $Z'_{1,g,n-1}(X, C)$ is simply connected. \square

Note that $H_2(Y_{g,n}; \mathbb{Z})$ has rank $4g(n-1)+2$, whereas the rank of $H_2(Y'_{1,g,n-1}; \mathbb{Z})$ is $4(g-1)(n-1)+2$. The point is that the fiber sum along $\Sigma_g = S_g$ contributes no rim tori classes (nor the associated

vanishing classes) because rim tori to the section S_g bound in $Y_{g,n-1} \setminus S_g$. (See the remark following Lemma 3.)

The fiber Σ_{n-1} of $Y_{g,n-1}$ has a basis for H_1 which is represented by the loops a_1, \dots, a_{2n-2} which were discussed in Section 2. Using the inclusion $\Sigma_{n-1} \subset S'$ and the identification of S' with C in $Z'_{1,g,n-1}(X, C)$ we obtain loops \bar{a}_i on $C \times \{\text{point}\}$ in $C \times \partial D^2 \subset X \setminus (C \times D^2)$. In the fiber sum with X , the rim tori to the \bar{a}_i and the vanishing classes contribute $2n - 2$ new hyperbolic pairs. Thus, $Z(X, C, g)$ and $Z'_{1,g,n-1}(X, C)$ have isomorphic intersection forms, and we get:

Proposition 7. *The symplectic manifolds $Z(X, C, g)$ and $Z'_{1,g,n-1}(X, C)$ are homeomorphic.*

Let $\beta' = (2n - 2)\Sigma_g + (2g - 2)S'$ be the canonical class of $Y'_{1,g,n-1}$. An argument similar to that of Section 2 shows that

$$SW_{Y'_{1,g,n-1}} = t_{\beta'} + t_{\beta'}^{-1} \text{ (+terms of lower degree in } t_{\Sigma_1} \text{ in case } g = 1).$$

We next restrict X to be a manifold which has the property that the above loops \bar{a}_i on $C \times \{\text{point}\}$ bound vanishing cycles (disks of self-intersection -1) in $X \setminus C$. This means that the boundary of the disk is allowed to move only in $C \times \{\text{point}\}$ when computing this self-intersection. For example, if we let X be the simply connected minimal elliptic surface without multiple fibers and with $\chi = n + 1$, $X = E(n + 1)$, and let C be a fiber of the genus n fibration on $E(n + 1)$, then the pair (X, C) will satisfy this hypothesis. When this hypothesis is satisfied we will say that $Y'_{1,g,n-1}$ and (X, C) are *complementary*.

Lemma 3 shows that the rim tori in $Y'_{1,g,n-1} \setminus S'$ which come from $H_1(S_1 \setminus \{\text{pt}\}; \mathbb{Z}) \rightarrow H_1(S'; \mathbb{Z})$ are nullhomologous in $Y' \setminus S'$. Let R_i denote the rim tori in $Z'_{1,g,n-1}(X, C)$ corresponding to the \bar{a}_i . If we assume that $Y'_{1,g,n-1}$ and (X, C) are complementary, then each \bar{a}_j bounds a disk of self-intersection -1 in $X \setminus C$. Now a_j lies on a fiber Σ_{n-1} in $Y_{g,n-1}$; so it lies in some copy of $S^1 \times M_{K_{n-1}}$. As we have seen in Section 2, a_j bounds a punctured torus of self-intersection $+1$ in this copy of $S^1 \times M_{K_{n-1}}$, and we can clearly make this punctured torus miss a section and a fiber of $S^1 \times M_{K_{n-1}}$. In $Z'_{1,g,n-1}(X, C)$, the (-1) -disk and the punctured torus glue together to give a torus U_j of self-intersection 0. The U_j satisfy

$$R_i \cdot U_j = \begin{cases} \pm 1, & j = i \pm 1, \\ 0, & j \neq i \pm 1. \end{cases}$$

Again as in Section 2, if κ is a basic class of X which satisfies $\kappa \cdot C = 2n - 2$ then we let β'_κ be the class represented by summing representatives of β' and κ along $C = S'$, and let $\epsilon'_\kappa = \beta'_\kappa + 2C$. Then [16] implies that

$$\sum_{r_1, \dots, r_n} SW_{Z'_{1,g,n-1}(X, C)} \left(\epsilon'_\kappa + \sum_i r_i R_i \right) = SW_{Y'_{1,g,n-1}}(\beta') \cdot SW_X(\kappa) = SW_X(\kappa)$$

and applying the adjunction inequality to the classes U_j shows that all $r_i = 0$. Thus, for basic classes κ of X which intersect C maximally,

$$SW_{Z'_{1,g,n-1}(X, C)}(\epsilon'_\kappa) = SW_X(\kappa).$$

Hence $Z'_{1,g,n-1}(X, C)$ has the same Seiberg–Witten (max) as $Z(X, C, g)$.

Proposition 8. *Let X be a symplectic 4-manifold containing an embedded symplectic surface C of genus $n \geq 2$ and self-intersection 0. Suppose also that $\pi_1(X \setminus C) = 0$ and that $Y'_{1,g,n-1}$ and (X, C) are complementary. Then $Z' = Z'_{1,g,n-1}(X, C)$ is simply connected, and if the Seiberg–Witten invariant of X is*

$$SW_X = SW_{\{X;C,-\}} + \sum_{k \cdot C = 2n-2} c_k(t_k + (-1)^{\chi(X)} t_k^{-1}),$$

then the Seiberg–Witten invariant of Z' is

$$SW_{Z'} = SW_{\{Z';C,-\}} + \sum_{k \cdot C = 2n-2} c_k(t_{e'_k} + (-1)^{\chi(Z')} t_{e'_k}^{-1}).$$

Whether $Z(X, C, 1)$ and $Z'_{1,g,n-1}(X, C)$ are in fact diffeomorphic is a very interesting question. More generally, let $L = \{k_1, \dots, k_n\}$ be a set of positive integers, and let

$$Y'_{1,g,L} = Y_{1,g} \#_{\Sigma_{g,i} = S_{g,i}} \prod_{i=1}^n Y_{g,k_i},$$

where $\Sigma_{g,i}$ are n genus g fibers of $Y_{1,g}$ and $S_{g,i}$ is the genus g section of Y_{g,k_i} . Then $Y'_{1,g,L}$ is a symplectic manifold which contains an embedded symplectic surface S' of genus $1 + \sum k_i$ and self-intersection number 0 formed from the sum of a section of $Y_{1,g}$ and fibers of the Y_{g,k_i} . The canonical class of $Y'_{1,g,L}$ is $\beta' = (2 \sum k_i) \Sigma_g + (2g - 2) S'$.

Now let X be a symplectic 4-manifold with an embedded symplectic surface C of genus $1 + \sum k_i$ and self-intersection 0. Then we can form

$$Z'_{1,g,L}(X, C) = X \#_{C=S'} Y'_{1,g,L}.$$

As above, if $X \setminus C$ is simply connected, then $Z'_{1,g,L}(X, C)$ is also simply connected. Arguments which are by now familiar show that

$$SW_{Y'_{1,g,n-1}} = t_{\beta'} + t_{\beta'}^{-1} (+ \text{terms of lower degree in } t_{\Sigma_g} \text{ in case } g = 1).$$

In $Z'_{1,g,L}(X, C)$, we can form classes β'_k and e'_k corresponding to the basic classes k of X which intersect C maximally. If κ is the canonical class of X then e'_κ is the canonical class of $Z'_{1,g,L}(X, C)$. There is an obvious extension of the definition of ‘complementarity’ for (X, C) and $Y'_{1,g,L}$.

Proposition 9. *Let $L = \{k_1, \dots, k_n\}$ be a set of positive integers, and let X be a symplectic 4-manifold containing an embedded symplectic surface C of genus $1 + \sum k_i$ and self-intersection 0. Suppose also that $\pi_1(X \setminus C) = 0$ and that (X, C) and $Y'_{1,g,L}$ are complementary. Then $Z' = Z'_{1,g,L}(X, C)$ is simply connected, and if the Seiberg–Witten invariant of X is*

$$SW_X = SW_{\{X;C,-\}} + \sum_{k \cdot C = 2n-2} c_k(t_k + (-1)^{\chi(X)} t_k^{-1}),$$

then the Seiberg–Witten invariant of Z' is

$$SW_{Z'} = SW_{\{Z';C,-\}} + \sum_{k \cdot C = 2n-2} c_k(t_{e'_k} + (-1)^{\chi(Z')} t_{e'_k}^{-1}).$$

As we noted above, the hypotheses of this proposition hold for X the elliptic surface $E(n)$, $n = 2 + \sum k_i$, and C a smooth fiber of its genus $(n - 1)$ -fibration.

5. How to make $Z'_{1,g,L}(X, C)$ nonsymplectic

Next, we show how to modify the manifolds $Z' = Z'_{1,g,L}(X, C)$ in order to manipulate the Seiberg–Witten invariant in such a way that it becomes impossible for the resulting manifold to admit a symplectic structure. We first identify families of nullhomologous tori in Z' with self-intersection number 0 upon which we will perform surgery. The manifold $Y_{1,g} = S^1 \times M_{K_g}$ is the total space of a fiber bundle $p : Y_{1,g} \rightarrow B$ with a genus 1 base and genus g fiber, Σ_g . Let S^1_1 and S^1_2 be embedded circles in the base which generate π_1 and such that the $p^{-1}(S^1_1) \cong S^1_1 \times \Sigma_g$ and $p^{-1}(S^1_2) \cong M_{K_g}$. We may suppose that S^1_1 and S^1_2 intersect in a single point, x_0 . For any closed embedded loop α in $p^{-1}(x_0)$, there is a torus $A(\alpha) = S^1_1 \times \alpha \subset Y_{1,g}$.

For the fiber sum $Y'_{1,g,L} = Y_{1,g} \#_{\Sigma_{g,i}=S_{g,i}} \prod_{i=1}^n Y_{g,k_i}$, let $b_i, i = 1, \dots, n$ be the points in B whose fibers Σ_{g,b_i} are identified with the sections $S_{g,i}$ of the Y_{g,k_i} . We may assume that $b_i \in B \setminus (S^1_1 \cup S^1_2)$ for each i , and so $A(\alpha)$ represents a class in $H_2(Y'_{1,g,L}; \mathbb{Z})$. This group is generated by Σ_g, S' , the rim tori and vanishing classes in the Y_{g,k_i} , and the rim tori and vanishing classes which arise from the Σ_{g,b_i} . However, since the Σ_{g,b_i} are identified with sections $S_{g,i}$ in $Y'_{1,g,L}$, these last rim tori are nullhomologous in $Y'_{1,g,L}$ and there are no corresponding vanishing classes. It is clear that $A(\alpha)$ is disjoint from all the rest of these surfaces; so this means that in $H_2(Y'_{1,g,L}; \mathbb{Z})$, $A(\alpha)$ is nullhomologous. We shall be interested in the case where $\alpha = a_i$, the loops giving the basis for $H_1(\Sigma_g; \mathbb{Z})$ which is described in Section 2.

We now wish to perform surgeries on collections of the $A(a_i)$. In $S^3 \setminus K_g$, a positive push-off a_i^+ of a_i bounds a punctured torus in the complement of K_g . Since $p^{-1}(S^1_2)$ is diffeomorphic to $M_{K_g} = (S^3 \setminus (K_g \times D^2)) \cup (S^1 \times D^2)$, we see that there is a push-off of a_i onto $\partial(a_i \times D^2)$ which bounds a punctured torus in $M_{K_g} \setminus (a_i \times D^2)$. Denote this push-off by \hat{a}_i . It defines a ‘0-framing’ for a_i , and we use it to express the boundary of a tubular neighborhood of $A(a_i)$ as $S^1_1 \times \hat{a}_i \times \partial D^2$. In general, remove this tubular neighborhood from Z and reglue it so that (in homology) ∂D^2 is sent to $m\hat{a}_i + \ell \partial D^2$ and S^1_1 to S^1_1 on $\partial(Z' \setminus (S^1_1 \times \hat{a}_i \times D^2))$. This gives ℓ/m -surgery on $A(a_i)$. We are interested in $(-1/m)$ -surgery for all $m \neq 0$.

For $m \neq 0$, $(-1/m)$ -surgery on a_1 in Fig. 2 turns K_g into a different knot $K_g(m)$ in S^3 . The construction gives an obvious genus g Seifert surface Σ for $K_g(m)$. (See Fig. 4.) It is then clear that $(-1/m)$ -surgery on $A(a_1)$ in $S^1 \times M_{K_g}$ gives $S^1 \times M_{K_g(m)}$. Performing this surgery on $A(a_1) \subset Y'_{1,g,L}$ gives us a new manifold

$$Y'_{1,g,L}(m) = S^1 \times M_{K_g(m)} \#_{\Sigma_{g,i}=S_{g,i}} \prod_{i=1}^n Y_{g,k_i}$$

and

$$Z'(m) = Z'_{1,g,L}(X, C; m) = X \#_{C=S'} Y'_{1,g,L}(m),$$

since S' is still defined using the torus $S^1 \times \{\text{meridian}\}$ in $S^1 \times M_{K_g(m)}$ and the fibers Σ_{k_i} in the Y_{g,k_i} . The argument of Proposition 6 shows that if $X \setminus C$ is simply connected, then so is $Z'(m)$.

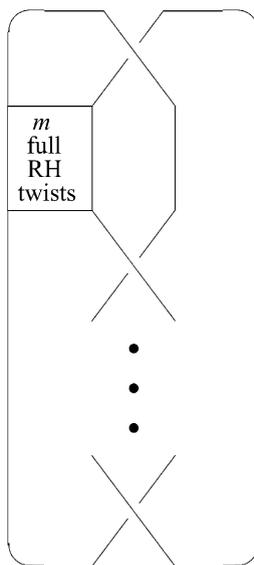


Fig. 4.

We next wish to calculate the Seiberg–Witten invariant of $Z'(m)$. Using the Mayer–Vietoris sequence for $Z' = (Z' \setminus (A(a_1) \times D^2)) \cup (A(a_1) \times D^2)$ and similarly for $Z'(m)$, we see that we may identify the homology groups of these manifolds. Let $A(a_1; m)$ be the core torus of $A(a_1) \times D^2$ in $Z'(m)$. Since we are performing $(-1/m)$ -surgery on the nullhomologous torus $A(a_1)$, the torus $A(a_1; m)$ is also nullhomologous. Let \hat{Z}' denote the result of the surgery on $A(a_1)$ for which ∂D^2 is sent to \hat{a}_1 and S_1^1 to S_1^1 on $\partial(Z' \setminus (S_1^1 \times \hat{a}_1 \times D^2))$. (This corresponds to $(0/1)$ -surgery on $A(a_1)$.) The intersection number of a basic class with a self-intersection 0 torus must be 0; so the basic classes of Z' may also be viewed as homology classes of $Z'(m)$ which have trivial intersection with the image torus $A(a_1; m)$. In \hat{Z}' there is the additional class τ which is represented by the torus $S_1^1 \times \{\text{point in } a_1\} \times \partial D^2$ on $\partial(Z' \setminus (A(a_1) \times D^2)) = \partial(Z' \setminus (S_1^1 \times a_1 \times D^2))$. Since $A(a_1)$ is nullhomologous, the surgery formula of Morgan et al. [15] and Taubes [22] (see also [7]) states

$$\text{SW}_{Z'(m)}(\zeta) = \text{SW}_{Z'}(\zeta) - m \sum_i \text{SW}_{Z'}(\zeta + i\tau).$$

For each basic class k of X which satisfies $k \cdot C = 2 \sum k_i$ we have classes β'_k and ε'_k in Z' . There are also such classes in $Z'(m)$, and to avoid further notational complexities, we shall still denote them by β'_k and ε'_k .

The knot $K_g(-1)$, obtained from $(+1)$ -surgery on a_1 , has a Seifert surface of genus $g - 1$, one less than that of K_g . Fig. 5 exhibits the case $g = 1$. In $M_{K_g(-1)}$ and therefore in $Y'_{1,g,L}(-1)$ there is a corresponding closed surface Γ_{g-1} . Calculating as in Section 2, we find that the only possible basic classes of $Y'_{1,g,L}(-1)$ are $\pm((2 \sum k_i)\Sigma_g + (2g - 2)S')$ (and lesser multiples of Σ_g in case $g = 1$). However, $S' \cdot \Gamma_{g-1} = 1$ and $\Sigma_g \cdot \Gamma_{g-1} = 0$. In case $g > 1$, the adjunction inequality, applied to Γ_{g-1} , leads directly to a contradiction if we assume that $\pm((2 \sum k_i)\Sigma_g + (2g - 2)S')$ is a basic class. In case $g = 1$, Γ_0 is a sphere of self-intersection 0, and it is essential since $S' \cdot \Gamma_0 = 1$ [6]. Thus, the

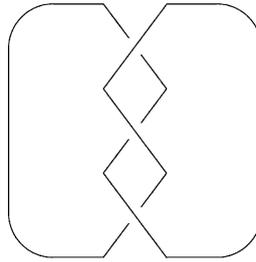


Fig. 5.

Seiberg–Witten invariant of $Y'_{1,g,L}(-1)$ vanishes in this case as well. Using [16], we get $SW_{Z'(-1)}=0$, and applying the above formula we obtain

$$0 = SW_{Z'}(\zeta) + \sum_i SW_{\hat{Z}'}(\zeta + i\tau)$$

for each basic class ζ of Z' , and hence:

Theorem 10. *Let $L=\{k_1, \dots, k_n\}$ be a set of positive integers, and let X be a symplectic 4-manifold containing an embedded symplectic surface C of genus $1 + \sum k_i$ and self-intersection 0. Suppose also that $\pi_1(X \setminus C)=0$ and that (X, C) and $Y'_{1,g,L}$ are complementary. Then for any positive integer m , $Z'(m) = Z'_{1,g,L}(X, C; m)$ is simply connected, and if the Seiberg–Witten invariant of X is*

$$SW_X = SW_{\{X;C,-\}} + \sum_{k \cdot C=2n-2} c_k (t_k + (-1)^{\chi(X)} t_k^{-1}),$$

then the Seiberg–Witten invariant of $Z'(m)$ is

$$SW_{Z'(m)} = SW_{\{Z'(m);C,-\}} + (m + 1) \sum_{k \cdot C=2n-2} c_k (t_{e'_k} + (-1)^{\chi(Z'(m))} t_{e'_k}^{-1}).$$

Of course, it follows that if $m \neq 0$, $Z'(m)$ can admit no symplectic structure.

It is clear that nothing special is gained by working with a_1 in this construction. If $\mathbf{m}=(m_1, \dots, m_{2g})$ where the m_i are nonnegative integers, and we perform $(-1/m_i)$ -surgeries on (push-offs) of the a_i in M_{K_g} we obtain 0-surgery on a new knot $K_g(\mathbf{m})$. Our construction then leads us to manifolds $Z'(\mathbf{m}) = Z'_{1,g,L}(X, C; \mathbf{m})$. As long as $X \setminus C$ is simply connected, $Z'(\mathbf{m})$ will also be simply connected, and, with notation as above,

Corollary 11. *The Seiberg–Witten invariant $SW_{\{Z'(\mathbf{m});C,\max\}}$ is*

$$SW_{\{Z'(\mathbf{m});C,\max\}} = \left(\prod_{i=1}^{2g} (m_i + 1) \right) \cdot \sum_{k \cdot C=2n-2} c_k (t_{e_k} + (-1)^{\chi(Z'(\mathbf{m}))} t_{e_k}^{-1}).$$

In the special case where X is the result of knot surgery on the K3-surface with a knot of genus $\sum k_i$ and C is the surface described in the paragraph below Proposition 4, (X, C) and $Y'_{1,g,L}$ are

complementary, and we get

$$SW_{Z'(\mathbf{m})} = \left(\prod_{i=1}^{2g} (m_i + 1) \right) \cdot (t_F^{n-1} \pm t_F^{1-n}).$$

So, for example, for any of the many choices of \mathbf{m} with corresponding products $\prod (m_i + 1)$ equal, we have manifolds which cannot be distinguished by means of their Seiberg–Witten invariants. One can also vary $L = \{k_1, \dots, k_n\}$.

For the record, we note following facts about the characteristic numbers of $Z'(\mathbf{m})$. ($\chi(Z'(\mathbf{m})) = \frac{1}{4}(\text{sign}(Z'(\mathbf{m})) + e(Z'(\mathbf{m})))$.)

Proposition 12. *The characteristic numbers of $Z'(\mathbf{m}) = Z'_{1,g,L}(X, C; \mathbf{m})$ are*

$$c_1^2(Z'(\mathbf{m})) = c_1^2(X) + 8 \left(n(g - 1) + \sum k_i \right)$$

and

$$\chi(Z'(\mathbf{m})) = \chi(X) + \left(n(g - 1) + \sum k_i \right).$$

6. Symplectic fibrations

A theorem of Parshin and Arakelov [2,17] (see also [13]) states that given fixed $g \geq 2$ and a finite set of points $S \subset \mathbf{CP}^1$, there are at most finitely many holomorphic fibrations over \mathbf{CP}^1 whose generic fiber is a Riemann surface of genus g and whose singular fibers have image in S . Interest in fibrations of symplectic manifolds has been rekindled by the theorem of Donaldson [5] which states that (after blowing up) each symplectic 4-manifold admits a locally holomorphic Lefschetz fibration over S^2 . In this section, we shall give examples which show that the Parshin–Arakelov Theorem has no analogue in the symplectic category (see Theorem 14). The authors have described more complicated examples exhibiting this phenomenon in [10].

The Lefschetz fibrations which we have in mind live on the homotopy elliptic surfaces of [7]. These manifolds, $E(n)_K$, are built from knot surgery on the simply connected, minimally elliptic surface $E(n)$ without multiple fibers and of holomorphic Euler characteristic n . If K is a nontrivial fibered knot, then $E(n)_K$ is a symplectic 4-manifold which admits no complex structure (nor does $E(n)_K$ with the opposite orientation).

The elliptic surface $E(n)$ is the double branched cover of $S^2 \times S^2$ with branch set equal to four disjoint copies of $S^2 \times \{\text{pt}\}$ together with $2n$ disjoint copies of $\{\text{pt}\} \times S^2$. The resultant branched cover has $8n$ singular points (corresponding to the double points in the branch set), whose neighborhoods are cones on \mathbf{RP}^3 . These are desingularized in the usual way, replacing their neighborhoods with cotangent bundles of S^2 . The result is $E(n)$. The horizontal and vertical fibrations of $S^2 \times S^2$ pull back to give fibrations of $E(n)$ over \mathbf{CP}^1 . A generic fiber of the vertical fibration is the double cover of S^2 , branched over four points—a torus. This describes an elliptic fibration of $E(n)$. The generic fiber of the horizontal fibration is the double cover of S^2 , branched over $2n$ points, and this gives a genus $n - 1$ fibration on $E(n)$. This genus $n - 1$ fibration has four singular fibers which are the preimages of the four $S^2 \times \{\text{pt}\}$'s in the branch set together with the spheres of self-intersection -2

arising from desingularization. The generic fiber T of the elliptic fibration meets a generic fiber Σ_{n-1} of the horizontal fibration in two points, $\Sigma_{n-1} \cdot T = 2$.

Let K be a fibered knot of genus g , and fix a generic elliptic fiber T_0 of $E(n)$. Then in the knot surgery manifold

$$E(n)_K = (E(n) \setminus (T_0 \times D^2)) \cup (S^1 \times (S^3 \setminus N(K))),$$

each normal 2-disk to T_0 is replaced by a fiber of the fibration of $S^3 \setminus N(K)$ over S^1 . Since T_0 intersects each generic horizontal fiber twice, we obtain a ‘horizontal’ fibration

$$h : E(n)_K \rightarrow \mathbf{CP}^1$$

of genus $2g + n - 1$.

This fibration also has four singular fibers arising from the four copies of $S^2 \times \{\text{pt}\}$ in the branch set of the double cover of $S^2 \times S^2$. Each of these gets blown up at $2n$ points in $E(n)$, and the singular fibers each consist of a genus g surface Σ_g of self-intersection $-n$ and multiplicity 2 with $2n$ disjoint 2-spheres of self-intersection -2 , each meeting Σ_g transversely in one point. The monodromy around each singular fiber is (conjugate to) the diffeomorphism of Σ_{2g+n-1} which is the deck transformation η of the double cover of Σ_g , branched over $2n$ points. Another way to describe η is to take the hyperelliptic involution ω of Σ_{n-1} and to connect sum copies of Σ_g at the two points of a nontrivial orbit of ω . Then ω extends to the involution η of Σ_{2g+n-1} .

The fibration which we have described is not Lefschetz since the singularities are not simple nodes. However, it can be perturbed locally to be Lefschetz.

Lemma 13. *Any symplectic fibration on a 4-manifold whose singular fibers are equivalent to those of $h : E(n)_K \rightarrow \mathbf{CP}^1$ can be locally deformed to a Lefschetz fibration.*

Proof. It suffices to find a holomorphic model for the singular fibers of h , since these can be locally deformed to a Lefschetz fibration by complex Morse theory. The model is built from a branched double cover of $\Sigma_g \times S^2$. This time the branch set consists of two disjoint copies of $\Sigma_g \times \{\text{pt}\}$ and $2n$ disjoint copies of $\{\text{pt}\} \times S^2$. After desingularizing as above, one obtains a complex surface $M(n, g)$ with a holomorphic (horizontal) fibration of genus $2g + n - 1$ and with a pair of singular fibers exactly of the type of the singular fibers of h . \square

Returning to our examples, we have:

Theorem 14. *If K is a fibered knot whose fiber has genus g , then $E(n)_K$ admits a locally holomorphic fibration (over \mathbf{CP}^1) of genus $2g + n - 1$ which has exactly four singular fibers. Furthermore, this fibration can be deformed locally to be Lefschetz.*

There are qualitative differences in the fibrations on $E(n)_K$ between $n = 1$ and $n > 1$. From their local description in Lemma 13, $E(1)_K$ contains singular fibers which are reducible, i.e. vanishing cycles obtained from Dehn twists about separating curves. This is due to the fact that it is built from a genus 0 fibration on $E(1)$. Siebert and Tian [19] have shown that any genus 2 Lefschetz fibration with only irreducible singular fibers must be holomorphic. Using this, Auroux [3] has

shown that any genus 2 Lefschetz fibration is stably holomorphic, i.e. the fiber sum of any genus 2 Lefschetz fibration with sufficiently many copies of the rational genus 2 Lefschetz fibration X_2 with 20 irreducible singular fibers is isomorphic to a holomorphic fibration. Thus, when K is either the trefoil or figure 8 knot (genus 1), then $E(1)_K$ admits a genus 2 Lefschetz fibration and $E(1)_K$ admits no complex structure. However, the fiber sum of $E(1)_K$ with four copies of X_2 is diffeomorphic to a complex surface. The case of $E(n)_K$, $n > 1$, is different. From their local description in Lemma 13, the fibrations in Theorem 14 are all of genus larger than 2 and have only irreducible singular fibers:

Proposition 15. *For $n \geq 2$, the genus $2g + n - 1$ Lefschetz fibrations, described above, on the manifolds $E(n)_K$ have no reducible singular fibers.*

Proof. Our argument determines the vanishing cycles of the Lefschetz fibration, and shows that they must all be nonseparating, or equivalently the corresponding singular fibers must be irreducible. It is easy to see that for any genus G Lefschetz fibration on a manifold X over S^2 , the euler number of X is given by $e(X) = s - 4G + 4$, where s is the number of singular fibers. Since $e(E(n)_K) = 12n$, our Lefschetz fibration has $16n + 8g - 8$ singular fibers. This fibration is obtained by a local deformation of a fibration ξ on $E(n)_K$ which has four singular fibers. Each of these four singular fibers contributes $4n + 2g - 2$ singular fibers and therefore $4n + 2g - 2$ vanishing cycles to the Lefschetz fibration.

By construction, the Lefschetz fibration on $E(n)_K$ is obtained from the standard genus $n - 1$ hyperelliptic Lefschetz fibration on $E(n)$. This fibration has $16n - 8$ singular fibers, and, for $n \geq 2$, it is well known (see e.g. [12]) that they are all irreducible.

We next take account of the fact that the singular fibers of ξ have genus g , whereas the nonsingular fibers have genus $2g + n - 1$. Vanishing cycles must account for the corresponding reduction of the first betti numbers of the fibers. Already we have taken into account the reduction in genus from $n - 1$ to 0, since this occurs in the hyperelliptic fibration. It remains to find $2g$ more vanishing cycles for each singular fiber of ξ . This accounts for the remaining $8g$ vanishing cycles, and they all must be nonseparating, since a separating vanishing cycle does not reduce the first homology of the fiber. \square

None of these examples, for $n \geq 2$, are hyperelliptic. Thus, while it is a conjecture of Siebert and Tian [19] that a hyperelliptic Lefschetz fibration with no reducible fibers is indeed holomorphic, the examples constructed in Theorem 14 show that hyperellipticity is a necessary hypothesis.

There is another way to view these constructions. Let $M(n, g)$ be the complex surface arising in the proof of Lemma 13. Once again, this manifold carries a pair of fibrations. There is a genus $2g + n - 1$ fibration over S^2 and an S^2 fibration over Σ_g .

Consider first the S^2 fibration. This has $2n$ singular fibers, each of which consists of a smooth 2-sphere E_i , $i = 1, \dots, 2n$, of self-intersection -1 and multiplicity 2, together with a pair of disjoint spheres of self-intersection -2 , each intersecting E_i once transversely. If we blow down E_i we obtain again an S^2 fibration over Σ_g , but the i th singular fiber now consists of a pair of 2-spheres of self-intersection -1 meeting once, transversely. Blowing down one of these gives another S^2 fibration over Σ_g , with one less singular fiber. Thus, blowing down $M(n, g)$ $4n$ times results in a manifold which is an S^2 bundle over Σ_g . This means that (if $n > 0$) $M(n, g)$ is diffeomorphic to $(S^2 \times \Sigma_g) \# 4n \overline{\mathbf{CP}}^2$.

The genus $2g + n - 1$ fibration on $M(n, g)$ has two singular fibers. As above, these fibers consist of a genus g surface Σ_g of self-intersection $-n$ and multiplicity 2 with $2n$ disjoint 2-spheres of self-intersection -2 , each meeting Σ_g transversely in one point. The monodromy of the fibration around each of these fibers is the deck transformation of the double branched cover of Σ_g . This is just the map η described above.

Let φ be a diffeomorphism of $\Sigma_g \setminus D^2$ which is the identity on the boundary. For instance, φ could be the monodromy of a fibered knot of genus g . There is an induced diffeomorphism Φ of $\Sigma_{2g+n-1} = \Sigma_g \# \Sigma_{n-1} \# \Sigma_g$ which is given by φ on the first Σ_g summand and by the identity on the other summands. Consider the twisted fiber sum

$$M(n, g) \#_{\varphi} M(n, g) = \{M(n, g) \setminus (D^2 \times \Sigma_{2g+n-1})\} \bigcup_{\text{id} \times \Phi} \{M(n, g) \setminus (D^2 \times \Sigma_{2g+n-1})\},$$

where fibered neighborhoods of generic fibers Σ_{2g+n-1} have been removed from the two copies of $M(n, g)$, and they have been glued by the diffeomorphism $\text{id} \times \Phi$ of $S^1 \times \Sigma_{2g+n-1}$.

In the case that φ is the monodromy of a fibered knot K , we claim that $M(n, g) \#_{\varphi} M(n, g)$ is the manifold $E(n)_K$ with the genus $2g + n - 1$ fibration described above. To see this, we view S^2 as the base of the horizontal fibration. Then it suffices to check that the total monodromy map $\pi_1(S^2 \setminus 4 \text{ points}) \rightarrow \text{Diff}(\Sigma_{2g+n-1})$ is the same for each. It is not difficult to see that if we write the generators of $\pi_1(S^2 \setminus 4 \text{ points})$ as α, β, γ with α and β representing loops around the singular points of, say, the image of the first copy of $M(n, g)$ and basepoint in this image, and γ a loop around a singular point in the image of the second $M(n, g)$ then the monodromy map μ satisfies $\mu(\alpha) = \eta$, $\mu(\beta) = \eta$ and $\mu(\gamma)$ is $\varphi \oplus \omega \oplus \varphi^{-1}$, expressed as a diffeomorphism of $\Sigma_g \# \Sigma_{n-1} \# \Sigma_g$. That this is also the monodromy of $E(n)_K$ follows directly from its construction.

7. Construction III

Let K_1, K_2 be fibered knots of genus g , and fix $n \geq 1$. Then as in the previous section there are genus $2g + n - 1$ fibrations on the manifolds $E(n)_{K_i}$. The manifolds $E(n)_{K_i}$ do not admit torus fibrations, but it makes sense to speak of ‘elliptic fibers’ T_i in $E(n)_{K_i}$, meaning those remaining from $E(n)$ after knot surgery. As above, T_i intersects the horizontal fiber Σ_{2g+n-1} in two positive intersection points.

Let $Y = Y(n; K_1, K_2)$ denote the fiber sum $E(n)_{K_1} \#_{\Sigma_{2g+n-1}} E(n)_{K_2}$. Then Y is a symplectic manifold with $c_1^2(Y) = 16g + 8n - 16$. Furthermore, Y is simply connected because in $E(n)_{K_i}$ a remnant of a singular fiber of the elliptic fibration on $E(n)$ provides a 2-sphere (of self-intersection -2) which intersects the fiber Σ_{2g+n-1} in one point. Tori $T_i \subset E(n)_{K_i}$, $i = 1, 2$ which intersect $\Sigma_{2g+n-1} \times S^1$ in identified circles will glue together to form a genus 3 surface in Y . Let τ denote the homology class in Y which is represented by this surface. Hence $\tau \cdot \Sigma_{2g+n-1} = 2$. The canonical class of Y is $K_Y = (2g + n - 2)\tau + 2\Sigma_{2g+n-1}$, and any basic class κ of Y satisfies $\kappa^2 = 16g + 8n - 16$.

Straightforward adjunction inequality arguments show that the only basic classes of Y are those of form $\pm K_Y + R$ where R lies in the subgroup of $H_2(Y; \mathbb{Z})$ generated by rim tori of Σ_{2g+n-1} . Let ω denote a symplectic form on Y obtained as a result of symplectic fiber sum. Then all rim tori of

Σ_{2g+n-1} are Lagrangian with respect to ω . Invoking [21], we see that $\int_{K_Y} \omega$ is the unique maximal value of $\int_{\kappa} \omega$ among all basic classes κ of Y . But because rim tori are Lagrangian

$$\int_{K_Y+R} \omega = \int_{K_Y} \omega.$$

This means that $K_Y + R$ is not basic if $R \neq 0$. Thus, the only basic classes of Y are $\pm K_Y$, and $\text{SW}_Y = t_K + (-1)^n t_K^{-1}$.

If $\{K_i\}$ is a family of genus g fibered knots, then the above discussion shows that manifolds $Y(n; K_i, K_j)$ cannot be distinguished on the basis of their Seiberg–Witten invariants.

References

- [1] S. Akbulut, Variations on Fintushel–Stern knot surgery on 4-manifolds, Proceedings of the Eighth Gokova Geometry-Topology Conference, 2001, pp. 1–12.
- [2] S. Arakelov, Families of algebraic curves with fixed degeneracies, *Izv. Akad. Nauk. SSSR Ser. Mat.* 35 (1971) 1269–1293.
- [3] D. Auroux, Fiber sums of genus 2 Lefschetz fibrations, *Turkish J. Math.* 27 (1) (2003) 1–10.
- [5] S. Donaldson, Lefschetz fibrations in symplectic geometry, Proceedings of the International Congress of Mathematicians, Vol. II, Berlin, 1998; Documents of Mathematics 1998, Extra Vol. II, 443–452.
- [6] R. Fintushel, R. Stern, Immersed spheres in 4-manifolds and the immersed Thom conjecture, *Turkish J. Math.* 19 (1995) 145–157.
- [7] R. Fintushel, R. Stern, Knots, links, and 4-manifolds, *Invent. Math.* 134 (1998) 363–400.
- [8] R. Fintushel, R. Stern, Constructions of smooth 4-manifolds, Proceedings of the International Congress of Mathematicians, Vol. II, Berlin, 1998, Documents of Mathematics 1998, Extra Vol. II, pp. 309–314.
- [9] R. Fintushel, R. Stern, Nonsymplectic 4-manifolds with one basic class, *Pacific J. Math.* 194 (2000) 325–333.
- [10] R. Fintushel, R. Stern, Symplectic surfaces in a fixed homology class, *J. Differential Geom.* 52 (2000) 203–222.
- [12] R. Gompf, A. Stipsicz, 4-Manifolds and Kirby Calculus, in: Graduate Studies in Mathematics, Vol. 20, American Mathematical Society, Providence, RI, 1999.
- [13] J. Jost, S-T. Yau, Applications of quasi-linear PDE to algebraic geometry and arithmetic lattices, in: Algebraic Geometry and Related Topics (Inchon, 1992), Conference Proceedings of Lecture Notes Algebraic Geometry, Vol. I, International Press, Boston, MA, 1993, pp. 169–193.
- [14] G. Meng, C. Taubes, $\text{SW} = \text{Milnor Torsion}$, *Math. Res. Lett.* 3 (1996) 661–674.
- [15] J. Morgan, T. Mrowka, Z. Szabo, Product formulas along T^3 for Seiberg–Witten invariants, *Math. Res. Lett.* 4 (1997) 915–929.
- [16] J. Morgan, Z. Szabo, C. Taubes, A product formula for the Seiberg–Witten invariants and the generalized Thom conjecture, *J. Differential Geom.* 44 (1996) 706–788.
- [17] A. Parshin, Algebraic curves over function fields, I, *Izv. Akad. Nauk. SSSR Ser. Mat.* 32 (1968) 1191–1219.
- [18] U. Persson, C. Peters, G. Xiao, Geography of spin surfaces, *Topology* 35 (1996) 845–862.
- [19] B. Siebert, G. Tian, On hyperelliptic C^∞ -Lefschetz fibrations of four-manifolds, *Commun. Contemp. Math.* 1 (1999) 255–280.
- [20] C. Taubes, The Seiberg–Witten invariants and symplectic forms, *Math. Res. Lett.* 1 (1994) 809–822.
- [21] C. Taubes, More constraints on symplectic manifolds from Seiberg–Witten invariants, *Math. Res. Lett.* 2 (1995) 9–14.
- [22] C. Taubes, The Seiberg–Witten invariants and 4-manifolds with essential tori, *Geom. Topol.* 5 (2001) 441–519.