Clay Mathematics Proceedings
Volume 5, 2006

Will We Ever Classify Simply-Connected Smooth 4-manifolds?

Ronald J. Stern

Abstract. These notes are adapted from two talks given at the 2004 Clay Institute Summer School on *Floer homology, gauge theory, and low dimensional topology* at the Alfred Rényi Institute. We will quickly review what we do and do not know about the existence and uniqueness of smooth and symplectic structures on closed, simply-connected 4-manifolds. We will then list the techniques used to date and capture the key features common to all these techniques. We finish with some approachable questions that further explore the relationship between these techniques and whose answers may assist in future advances towards a classification scheme.

1. Introduction

Despite spectacular advances in defining invariants for simply-connected smooth and symplectic 4-dimensional manifolds and the discovery of important qualitative features about these manifolds, we seem to be retreating from any hope to classify simply-connected smooth or symplectic 4-dimensional manifolds. The subject is rich in examples that demonstrate a wide variety of disparate phenomena. Yet it is precisely this richness which, at the time of these lectures, gives us little hope to even conjecture a classification scheme. In these notes, adapted from two talks given at the 2004 Clay Institute Summer School on *Floer homology, gauge theory, and low dimensional topology* at the Alfred Rényi Institute, we will quickly review what we do and do not know about the existence and uniqueness of smooth and symplectic structures on closed, simply-connected 4-manifolds. We will then list the techniques used to date and capture the key features common to all these techniques. We finish with some approachable questions that further explore the relationship between these techniques and whose answers may assist in future advances towards a classification scheme.

Algebraic Topology. The critical algebraic topological information for a closed, simply-connected, smooth 4-manifold X is encoded in its Euler characteristic $e(X)$, its signature $\sigma(X)$, and its type $t(X)$ (either 0 if the intersection form of X is even and 1 if it is odd). These invariants completely classify the homeomorphism

1991 *Mathematics Subject Classification.* Primary 57R55, 57R57, 14J26; Secondary 53D05.
Key words and phrases. 4-manifold, Seiberg-Witten invariant, symplectic, Lagrangian.
The author was partially supported by NSF Grant DMS0204041.

type of X ([3, 12]). We recast these algebraic topological invariants by defining $\chi_h(X) = (e(X) + \sigma(X))/4$, which is the holomorphic Euler characteristic in the case that X is a complex surface, and $c(X) = 3\sigma(X) + 2e(X)$, which is the self-intersection of the first Chern class of X in the case that X is complex.

Analysis. To date, the critical analytical information for a smooth, closed, simply-connected 4-manifold X is encoded in its Seiberg-Witten invariants [30]. When $\chi_h(X) > 1$ this integer-valued function SW_X is defined on the set of *spin*^{c} structures over X. Corresponding to each *spin*^{c} structure $\mathfrak{s}$ over X is the bundle of positive spinors $W^+_{\mathfrak{s}}$ over X. Set $c(\mathfrak{s}) \in H_2(X)$ to be the Poincaré dual of $c_1(W^+_{\mathfrak{s}})$. Each $c(\mathfrak{s})$ is a characteristic element of $H_2(X;\mathbf{Z})$ (i.e. its Poincaré dual $\hat{c}(\mathfrak{s}) = c_1(W^+_{\mathfrak{s}})$ reduces mod 2 to $w_2(X)$). The sign of SW_X depends on a homology orientation of X, that is, an orientation of $H^0(X;\mathbf{R}) \otimes \det H^2_+(X;\mathbf{R}) \otimes \det H^1(X;\mathbf{R})$. If $\mathrm{SW}_X(\beta) \neq 0$, then β is called a *basic class* of X. It is a fundamental fact that the set of basic classes is finite. Furthermore, if β is a basic class, then so is $-\beta$ with $\mathrm{SW}_X(-\beta) = (-1)^{\chi_h(X)}\,\mathrm{SW}_X(\beta)$. The Seiberg-Witten invariant is an orientation-preserving diffeomorphism invariant of X (together with the choice of a homology orientation). We recast the Seiberg-Witten invariant as an element of the integral group ring $\mathbf{Z}H_2(X)$, where for each $\alpha \in H_2(X)$ we let t_α denote the corresponding element in $\mathbf{Z}H_2(X)$. Suppose that $\{\pm\beta_1, \ldots, \pm\beta_n\}$ is the set of nonzero basic classes for X. Then the Seiberg-Witten invariant of X is the Laurent polynomial

$$\mathcal{SW}_X = \mathrm{SW}_X(0) + \sum_{j=1}^{n} \mathrm{SW}_X(\beta_j) \cdot (t_{\beta_j} + (-1)^{\chi_h(X)}\, t^{-1}_{\beta_j}) \in \mathbf{Z}H_2(X).$$

When $\chi_h = 1$ the Seiberg-Witten invariant depends on a given orientation of $H^2_+(X;\mathbf{R})$, a given metric g, and a self-dual 2-form as follows. There is a unique g-self-dual harmonic 2-form $\omega_g \in H^2_+(X;\mathbf{R})$ with $\omega_g^2 = 1$ and corresponding to the positive orientation. Fix a characteristic homology class $k \in H_2(X;\mathbf{Z})$. Given a pair (A, ψ), where A is a connection in the complex line bundle whose first Chern class is the Poincaré dual $\widehat{k} = \frac{i}{2\pi}[F_A]$ of k and ψ a section of the bundle W^+ of self-dual spinors for the associated *spin*^{c} structure, the perturbed Seiberg-Witten equations are:

$$D_A\psi = 0$$
$$F_A^+ = q(\psi) + i\eta$$

where F_A^+ is the self-dual part of the curvature F_A, D_A is the twisted Dirac operator, η is a self-dual 2-form on X, and q is a quadratic function. Write $\mathrm{SW}_{X,g,\eta}(k)$ for the corresponding invariant. As the pair (g, η) varies, $\mathrm{SW}_{X,g,\eta}(k)$ can change only at those pairs (g, η) for which there are solutions with $\psi = 0$. These solutions occur for pairs (g, η) satisfying $(2\pi\widehat{k} + \eta) \cdot \omega_g = 0$. This last equation defines a wall in $H^2(X;\mathbf{R})$.

The point ω_g determines a component of the double cone consisting of elements of $H^2(X;\mathbf{R})$ of positive square. We prefer to work with $H_2(X;\mathbf{R})$. The dual component is determined by the Poincaré dual H of ω_g. (An element $H' \in H_2(X;\mathbf{R})$ of positive square lies in the same component as H if $H' \cdot H > 0$.) If $(2\pi\widehat{k} + \eta) \cdot \omega_g \neq 0$ for a generic η, $\mathrm{SW}_{X,g,\eta}(k)$ is well-defined, and its value depends only on the sign of $(2\pi\widehat{k} + \eta) \cdot \omega_g$. Write $\mathrm{SW}^+_{X,H}(k)$ for $\mathrm{SW}_{X,g,\eta}(k)$ if $(2\pi\widehat{k} + \eta) \cdot \omega_g > 0$ and $\mathrm{SW}^-_{X,H}(k)$ in the other case.

The invariant $\mathrm{SW}_{X,H}(k)$ is defined by $\mathrm{SW}_{X,H}(k) = \mathrm{SW}^+_{X,H}(k)$ if $(2\pi\widehat{k})\cdot\omega_g > 0$, or dually, if $k \cdot H > 0$, and $\mathrm{SW}_{X,H}(k) = \mathrm{SW}^-_{X,H}(k)$ if $H \cdot k < 0$. The wall-crossing formula [15, 16] states that if H', H'' are elements of positive square in $H_2(X;\mathbf{R})$ with $H' \cdot H > 0$ and $H'' \cdot H > 0$, then if $k \cdot H' < 0$ and $k \cdot H'' > 0$,

$$\mathrm{SW}_{X,H''}(k) - \mathrm{SW}_{X,H'}(k) = (-1)^{1+\frac{1}{2}d(k)}$$

where $d(k) = \frac{1}{4}(k^2 - (3\,\mathrm{sign} + 2\,e)(X))$ is the formal dimension of the Seiberg-Witten moduli spaces.

Furthermore, in case $b^- \leq 9$, the wall-crossing formula, together with the fact that $\mathrm{SW}_{X,H}(k) = 0$ if $d(k) < 0$, implies that $\mathrm{SW}_{X,H}(k) = \mathrm{SW}_{X,H'}(k)$ for any H' of positive square in $H_2(X;\mathbf{R})$ with $H \cdot H' > 0$. So in case $b^+_X = 1$ and $b^-_X \leq 9$, there is a well-defined Seiberg-Witten invariant, $\mathrm{SW}_X(k)$.

Possible Classification Schemes. From this point forward and unless otherwise stated all manifolds will be closed and simply-connected. In order to avoid trivial constructions we consider *irreducible* manifolds, i.e. those that cannot be represented as the connected sum of two manifolds except if one factor is a homotopy 4-sphere. (We still do not know if there exist smooth homotopy 4-spheres not diffeomorphic to the standard 4-sphere S^4).

So the existence part of a classification scheme for irreducible smooth (symplectic) 4-manifolds could take the form of determining which $(\chi_h, c, t) \in \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}_2$ can occur as $(\chi_h(X), c(X), t(X))$ for some smooth (symplectic) 4-manifold X. This is referred to as *the geography problem*. The game plan would be to create techniques to realize all possible lattice points. The uniqueness part of the classification scheme would then be to determine all smooth (symplectic) 4-manifolds with a fixed $(\chi_h(X), c(X), t(X))$ and determine invariants that would distinguish them. Again, the game plan would be to create techniques that preserve the homeomorphism type yet change these invariants.

In the next two sections we will outline what is and is not known about the existence (geography) and uniqueness problems without detailing the techniques. Then we will list the techniques used, determine their interplay, and explore questions that may yield new insight. A companion approach, which we will also discuss towards the end of these lectures, is to start with a particular well-understood class of 4-manifolds and determine how all other smooth (symplectic) 4-manifolds can be constructed from these.

2. Existence

Our current understanding of the geography problem is given by Figure 1 where all known simply-connected smooth irreducible 4-manifolds are plotted as lattice points in the (χ_h, c)-plane. In particular, all known simply-connected irreducible smooth or symplectic 4-manifolds have $0 \leq c < 9\chi_h$ and every lattice point in that region can be realized by a symplectic (hence smooth) 4-manifold.

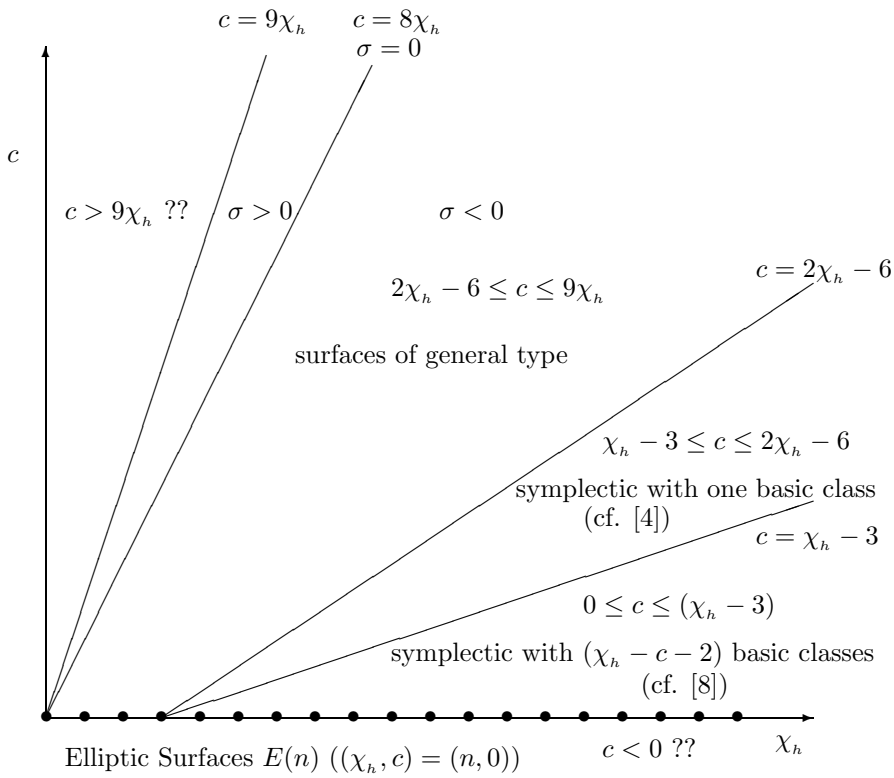


Figure 1

An irreducible 4-manifold need not lie on a lattice point. The issue here is whether $\chi_h \in \mathbf{Z}$ or $\chi_h \in \mathbf{Z}[\frac{1}{2}]$. Note that $\chi_h(X) \in \mathbf{Z}$ iff X has an almost-complex structure. In addition, the Seiberg-Witten invariants are only defined for manifolds with $\chi_h \in \mathbf{Z}$. Since our only technique to determine if a 4-manifold is irreducible is to use the fact that the Seiberg-Witten invariants of a reducible 4-manifold vanish, all known irreducible 4-manifolds have $\chi_h \in \mathbf{Z}$.

Problem 1. *Do there exist irreducible smooth 4-manifolds with $\chi_h \notin \mathbf{Z}$?*

Here the work of Bauer and Furuta [2] on stable homotopy invariants derived from the Seiberg-Witten equations may be useful. To expose our ignorance, consider two copies of the elliptic surface $E(2)$. Remove the neighborhood of a sphere with self-intersection -2 from each and glue together the resulting manifolds along their boundary $\mathbf{RP}^3$ using the orientation reversing diffeomorphism of $\mathbf{RP}^3$. The result has $\chi_h \notin \mathbf{Z}$ and it is unknown if it is irreducible.

All complex manifolds with $c = 9\chi_h > 9$ are non-simply-connected, in particular they are ball quotients. Thus obvious problems are:

Problem 2. *Do there exist irreducible simply connected smooth or symplectic manifolds with $c = 9\chi_h > 9$?*

PROBLEM 3. *Does there exist an irreducible non-complex smooth or symplectic manifold X with $\chi_h > 1$, $c = 9\chi_h$ (with any fundamental group), $\mathcal{SW}_X \neq 0$, and which is not a ball-quotient?*

PROBLEM 4. *Do there exist irreducible smooth or symplectic manifolds with $c > 9\chi_h$?*

The work of Taubes [28] on the relationship between Seiberg-Witten and Gromov-Witten invariants shows that $c \geq 0$ for an irreducible symplectic 4-manifold.

PROBLEM 5. *Do there exist simply connected irreducible smooth manifolds with $c < 0$?*

There appears to be an interesting relationship between the number of Seiberg-Witten basic classes and the pair (χ_h, c). In particular, all known smooth 4-manifolds with $0 \leq c \leq \chi_h - 3$ have at least $\chi_h - c - 2$ Seiberg-Witten basic classes [4]. So

PROBLEM 6. *Does there exist an irreducible smooth manifold X with $0 \leq c(X) \leq \chi_h(X) - 3$ and with fewer than $\chi_h(X) - c(X) - 2$ Seiberg-Witten basic classes? (There is a physics proof that there are no such examples [17].)*

Figure 1 contains no information about the geography of spin 4-manifolds, i.e. manifolds with $t = 0$. For a spin 4-manifold there is the relation $c = 8\chi_h \mod 16$. Almost every lattice point with $c = 8\chi_h \mod 16$ and $0 \leq c < 9\chi_h$ can be be realized by an irreducible spin 4-manifold [21]. Surprisingly not all of the lattice points with $2\chi_h \leq 3(\chi_h - 5)$ can be realized by complex manifolds with $t = 0$ [24], so spin manifolds with $2\chi_h \leq 3(\chi_h - 5)$ provide several examples of smooth irreducible 4-manifolds with $2\chi_h - 6 \leq c < 9\chi_h$ that support no complex structure (cf. [9]). There remains a better understanding of manifolds close to the $c = 9\chi_h$ line, in particular those with $9\chi_h > c \geq 8.76\chi_h$ and not on the lines $c = 9\chi_h - k$ with $k \leq 121$ (cf. [24]).

The techniques used in all these constructions are an artful application of the generalized fiber sum construction (cf. [13]) and the rational blowdown construction [6], which we will discuss later in this lecture.

3. Uniqueness

Here is where we begin to lose control of the classification of smooth 4-manifolds. If a topological 4-manifold admits an irreducible smooth (symplectic) structure that has a smoothly (symplectically) embedded torus with self-intersection zero and with simply-connected complement, then it also admits infinitely many distinct smooth (symplectic) structures and also admits infinitely many distinct smooth structures with no compatible symplectic structure. The basic technique here is the knot-surgery construction of Fintushel-Stern [7], i.e. remove a neighborhood $T^2 \times D^2 = S^1 \times S^1 \times D^2$ of this torus and replace it with $S^1 \times S^3 \setminus K$ where K is a knot in S^3. As we will point out later, the resulting smooth structures are distinguished by the Alexander polynomial of the knot K. There are no known examples of (simply-connected) smooth or symplectic 4-manifolds with $\chi_h > 1$ that do not admit such tori. Hence, there are no known smooth or symplectic 4-manifolds with $\chi_h > 1$ that admit finitely many smooth or symplectic structures. Thus,

PROBLEM 7. *Do there exist irreducible smooth (symplectic) 4-manifolds with $\chi_h > 1$ that do not admit a smoothly (symplectically) embedded torus with self-intersection 0 and simply-connected complement?*

All of the constructions used for the geography problem with $\chi_h > 1$ naturally contain such tori, so the only hope is to find manifolds where these constructions have yet to work, i.e. those with $8.76 < c \leq 9\chi_h$, that do not contain such tori.

PROBLEM 8. *Do manifolds with $c = 9\chi_h$ admit exotic smooth structures?*

The situation for $\chi_h = 1$ is potentially more interesting and may yield phenomena not shared by manifolds with $\chi_h > 1$. For example, the complex projective plane $\mathbf{CP}^2$ has $c = 9\chi_h = 9$ and is simply-connected. It is also known that $\mathbf{CP}^2$ as a smooth manifold has a unique symplectic structure [27,28]. Thus, a fundamental question that still remains is

PROBLEM 9. *Does the complex projective plane $\mathbf{CP}^2$ admit exotic smooth structures?*

PROBLEM 10. *What is the smallest m for which $\mathbf{CP}^2 \# m\,\overline{\mathbf{CP}}^2$ admits an exotic smooth structure?*

The primary reason for our ignorance here is that for $c > 1$ (i.e. $m < 9$), these manifolds do not contain homologically essential tori with zero self-intersction. Since the rational elliptic surface $E(1) \cong \mathbf{CP}^2 \# 9\,\overline{\mathbf{CP}}^2$ admits tori with self-intersection zero, it has infinitely many distinct smooth structures. In the late 1980's Dieter Kotschick [14] proved that the Barlow surface, which was known to be homeomorphic to $\mathbf{CP}^2 \# 8\overline{\mathbf{CP}}^2$, is not diffeomorphic to it. In the following years the subject of simply connected smooth 4-manifolds with $m < 8$ languished because of a lack of suitable examples. However, largely due to a beautiful paper of Jongil Park [22], who found the first examples of exotic simply connected 4-manifolds with $m = 7$, interest was revived. Shortly after this conference ended, Peter Ozsváth and Zoltán Szabó proved that Park's manifold is minimal [20] by computing its Seiberg-Witten invariants. Then András Stipsicz and Zoltán Szabó used a technique similar to Park's to construct an exotic manifold with $m = 6$ [25]. The underlying technique in these constructions is an artful use of the rational blowdown construction.

Since $\mathbf{CP}^2 \# m\,\overline{\mathbf{CP}}^2$ for $m < 9$ does not contain smoothly embedded tori with self-intersection zero, it has not been known whether it can have an infinite family of smooth structures. Most recently, Fintushel and Stern [11], introduced a new technique which was used to show that for $6 \leq m \leq 8$, $\mathbf{CP}^2 \# m\,\overline{\mathbf{CP}}^2$ does indeed have an infinite family of smooth structures, and, in addition, none of these smooth structures support a compatible symplectic structure. These are the first examples of manifolds that do not contain homologically essential tori, yet have infinitely many distinct smooth structures. Park, Stipsicz, and Szabó [23], and independently Fintushel and Stern [11] used this construction to show that $m = 5$ also has an infinite family of smooth structures none of which support a compatible symplectic structure (cf. [11]). The basic technique in these constructions is a prudent blend of the knot surgery and rational blowdown constructions.

As is pointed out in [25], the Seiberg-Witten invariants will never distinguish more than two distinct irreducible symplectic structures on $\mathbf{CP}^2 \# m\,\overline{\mathbf{CP}}^2$ for $m <$

9. Basically, this is due to the fact that if there is more than one pair of basic classes for a $\chi_h = 1$ manifold, then it is not minimal. So herein lies one of our best hopes for finiteness in dimension 4.

PROBLEM 11. *Does* $\mathbf{CP}^2 \# m\,\overline{\mathbf{CP}}^2$ *for* $m < 9$ *support more than two irreducible symplectic structures that are not deformation equivalent?*

4. The techniques used for the construction of all known simply-connected smooth and symplectic 4-manifolds

The construction of simply-connected smooth or symplectic 4-manifolds sometimes takes the form of art rather than science. This is exposed by the lack of success in proving structural theorems or uncovering any finite phenomena in dimension 4. In this lecture we will list all the constructions used in building the 4-manifolds necessary for the results of the first two sections and try to bring all the unusual phenomena in dimension 4 into a framework that will allow us to at least understand those surgical operations that one performs to go from one smooth structure on a given simply-connected 4-manifold to any other smooth structure. This will take the form of understanding a variety of cobordisms between 4-manifolds.

Here is the list of constructions used in the first two sections.

generalized fiber sum: Assume two 4-manifolds X_1 and X_2 each contain an embedded genus g surface $F_j \subset X_j$ with self-intersection 0. Identify tubular neighborhoods νF_j of F_j with $F_j \times D^2$ and fix a diffeomorphism $f : F_1 \to F_2$. Then the fiber sum $X = X_1 \#_f X_2$ of (X_1, F_1) and (X_2, F_2) is defined as $X_1 \setminus \nu F_1 \cup_\phi X_2 \setminus \nu F_2$, where ϕ is $f\times$ (complex conjugation) on the boundary $\partial(X_j \setminus \nu F_j) = F_j \times S^1$.

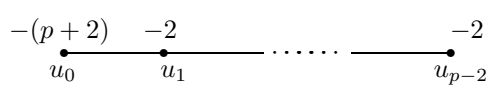
generalized logarithmic transform: Assume that X contains a homologically essential torus T with self-intersection zero. Let νT denote a tubular neighborhood of T. Deleting the interior of νT and regluing $T^2 \times D^2$ via a diffeomorphism $\phi : T^2 \times D^2 \to \partial(X - \text{int } \nu T) = \partial \nu T$ we obtain a new manifold X_ϕ, the generalized logarithmic transform of X along T.

If p denotes the absolute value of the degree of the map $\pi \circ \phi : \{pt\} \times S^1 \to \pi(\partial \nu T) = S^1$, then X_ϕ is called a generalized logarithmic transformation of multiplicity p.

If the complement of T is simply-connected and $t(X) = 1$, then X_ϕ is homeomorphic to X. If the complement of T is simply-connected and $t(X) = 0$, then X_ϕ is homeomorphic to X if p is odd, otherwise X_ϕ has the same c and χ_h but with $t(X_\phi) = 1$.

blowup: Form $X \# \overline{\mathbf{CP}}^2$.

rational blowdown : Let C_p be the smooth 4-manifold obtained by plumbing $(p-1)$ disk bundles over the 2-sphere according to the diagram



Then the classes of the 0-sections have self-intersections $u_0^2 = -(p+2)$ and $u_i^2 = -2$, $i = 1, \ldots, p-2$. The boundary of C_p is the lens space $L(p^2, 1 -$

p) which bounds a rational ball B_p with $\pi_1(B_p) = \mathbf{Z}_p$ and $\pi_1(\partial B_p) \to \pi_1(B_p)$ surjective. If C_p is embedded in a 4-manifold X then the rational blowdown manifold $X_{(p)}$ is obtained by replacing C_p with B_p, i.e., $X_{(p)} = (X \setminus C_p) \cup B_p$ (cf. [6]). If $X \setminus C_p$ is simply connected, then so is $X_{(p)}$

knot surgery: Let X be a 4-manifold which contains a homologically essential torus T of self-intersection 0, and let K be a knot in S^3. Let $N(K)$ be a tubular neighborhood of K in S^3, and let $T \times D^2$ be a tubular neighborhood of T in X. Then the knot surgery manifold X_K is defined by

$$X_K = (X \setminus (T \times D^2)) \cup (S^1 \times (S^3 \setminus N(K))$$

The two pieces are glued together in such a way that the homology class $[\mathrm{pt} \times \partial D^2]$ is identified with $[\mathrm{pt} \times \lambda]$ where λ is the class of a longitude of K. If the complement of T in X is simply-connected, then X_K is homeomorphic to X.

The Seiberg-Witten invariants are sensitive to all of these operations.

generalized logarithmic transform: If T is contained in a node neighborhood, then

$$\mathcal{SW}_{X_\phi} = \mathcal{SW}_X \cdot (s^{-(p-1)} + s^{-(p-3)} + \cdots + s^{(p-1)})$$

where $s = \exp(T/p)$, p the order of the generalized logarithmic transform (cf. [6]).

blowup: The relationship between the Seiberg-Witten invariants of X and its blowup $X\#\overline{\mathbf{CP}}^2$ is referred to as the blowup formula and was given in Witten's original article [30] (cf. [5]). In particular, if e is the homology class of the exceptional curve and $\{B_1, \ldots, B_n\}$ are the basic classes of X, then the basic classes of $X\#\overline{\mathbf{CP}}^2$ are $\{B_1 \pm E, \ldots, B_n \pm E\}$ and $\mathrm{SW}_{X\#\overline{\mathbf{CP}}^2}(B_j \pm E) = \mathrm{SW}_X(B_j)$.

rational blowdown: The Seiberg-Witten invariants of X and $X_{(p)}$ can be compared as follows. The homology of $X_{(p)}$ can be identified with the orthogonal complement of the classes u_i, $i = 0, \ldots, p-2$ in $H_2(X; \mathbf{Z})$, and then each characteristic element $k \in H_2(X_{(p)}; \mathbf{Z})$ has a lift $\widetilde{k} \in H_2(X; \mathbf{Z})$ which is characteristic and for which the dimensions of moduli spaces agree, $d_{X_{(p)}}(k) = d_X(\widetilde{k})$. It is proved in [6] that if $b_X^+ > 1$ then $\mathrm{SW}_{X_{(p)}}(k) = \mathrm{SW}_X(\widetilde{k})$. In case $b_X^+ = 1$, if $H \in H_2^+(X; \mathbf{R})$ is orthogonal to all the u_i then it also can be viewed as an element of $H_2^+(X_{(p)}; \mathbf{R})$, and $\mathrm{SW}_{X_{(p)},H}(k) = \mathrm{SW}_{X,H}(\widetilde{k})$.

knot surgery: If, for example, T is contained in a node neighborhood and $\chi_h(X) > 1$ then the Seiberg-Witten invariant of the knot surgery manifold X_K is given by

$$\mathcal{SW}_{X_K} = \mathcal{SW}_X \cdot \Delta_K(t)$$

where $\Delta_K(t)$ is the symmetrized Alexander polynomial of K and $t = \exp(2[T])$. When $\chi_h = 1$, the Seiberg-Witten invariants of X_K are still completely determined by those of X and the Alexander polynomial $\Delta_K(t)$ [7].

Here T contained in a node neighborhood means that an essential loop on $\partial\nu T$ bounds a disk in the complement with relative self-intersection -1. We sometimes refer to this disk as a vanishing cycle.

In many circumstances, there are formulas for determining the Seiberg-Witten invariants of a fiber sum in terms of the Seiberg-Witten invariants of X_1 and X_2 and how the basic classes intersect the surfaces F_1 and F_2.

Interaction of the operations. While knot surgery appears to be a new operation, the constructions in [7] point out that the knot surgery construction is actually a series of ± 1 generalized logarithmic transformations on *null-homologous tori*. To see this, note that any knot can be unknotted via a sequence of crossing changes, which in turn can be realized as a sequence of ± 1 surgeries on unknotted curves $\{c_1, \ldots, c_n\}$ that link the knot algebraically zero times and geometrically twice. When crossed with S^1 this translates to the fact that X can be obtained from X_K via a sequence of ± 1 generalized logarithmic transformations on the *null-homologous tori* $\{S^1 \times c_1, \ldots, S^1 \times c_n\}$ in X_K. So the hidden mechanism behind the knot surgery construction is generalized logarithmic transformations on null-homologous tori. The calculation of the Seiberg-Witten invariants is then reduced to understanding how the Seiberg-Witten invariants change under a generalized logarithmic transformation on a null-homologous torus. This important formula is due to Morgan, Mrowka, and Szabó [18] (see also [29]). For this formula fix simple loops α, β, δ on $\partial N(T)$ whose homology classes generate $H_1(\partial N(T))$. If $\omega = p\alpha + q\beta + r\delta$ write $X_T(p,q,r)$ instead of $X_T(\omega)$. Given a class $k \in H_2(X)$:

$$\begin{aligned}(1)\quad \sum_i \mathrm{SW}_{X_T(p,q,r)}(k_{(p,q,r)} + 2i[T]) &= p\sum_i \mathrm{SW}_{X_T(1,0,0)}(k_{(1,0,0)} + 2i[T]) + \\ &+ q\sum_i \mathrm{SW}_{X_T(0,1,0)}(k_{(0,1,0)} + 2i[T]) + r\sum_i \mathrm{SW}_{X_T(0,0,1)}(k_{(0,0,1)} + 2i[T])\end{aligned}$$

In this formula, T denotes the torus which is the core $T^2 \times 0 \subset T^2 \times D^2$ in each specific manifold $X(a,b,c)$ in the formula, and $k_{(a,b,c)} \in H_2(X_T(a,b,c))$ is any class which agrees with the restriction of k in $H_2(X \setminus T \times D^2, \partial)$ in the diagram:

$$\begin{array}{ccc} H_2(X_T(a,b,c)) & \longrightarrow & H_2(X_T(a,b,c), T \times D^2) \\ & & \downarrow \cong \\ & & H_2(X \setminus T \times D^2, \partial) \\ & & \uparrow \cong \\ H_2(X) & \longrightarrow & H_2(X, T \times D^2) \end{array}$$

Let $\pi(a,b,c) : H_2(X_T(a,b,c)) \to H_2(X \setminus T \times D^2, \partial)$ be the composition of maps in the above diagram, and $\pi(a,b,c)_*$ the induced map of integral group rings. Since we are often interested in invariants of the pair (X,T), it is sometimes useful to work with

$$\overline{\mathcal{SW}}_{(X_T(a,b,c),T)} = \pi(a,b,c)_*(\mathcal{SW}_{X_T(a,b,c)}) \in \mathbf{Z}H_2(X \setminus T \times D^2, \partial).$$

The indeterminacy due to the sum in (1) is caused by multiples of $[T]$; so passing to $\overline{\mathcal{SW}}$ removes this indeterminacy, and the Morgan-Mrowka-Szabó formula becomes

$$(2)\quad \overline{\mathcal{SW}}_{(X_T(p,q,r),T)} = p\overline{\mathcal{SW}}_{(X_T(1,0,0),T)} + q\overline{\mathcal{SW}}_{(X_T(0,1,0),T)} + r\overline{\mathcal{SW}}_{(X_T(0,0,1),T)}.$$

So if we expand the notion of generalized logarithmic transformation to include both homologically essential and null-homologous tori, then we can eliminate the knot surgery construction from our list of essential surgery operations. Thus our list is of essential operations is reduced to

- generalized fiber sum
- generalized logarithmic transformations on a torus with trivial normal bundle
- blowup
- rational blowdown

There are further relationships between these operations. In [6] it is shown that if T is contained in a node neighborhood, then a generalized logarithmic transformation can be obtained via a sequence of blowups and rational blowdowns. (This together with work of Margaret Symington [26] shows that logarithmic transformations ($p \neq 0$) on a symplectic torus results in a symplectic manifold. We do not know of any other proof that a generalized logarithmic transformations on a symplectic torus in a node neighborhood results in a symplectic manifold.) However, it is not clear that a rational blowdown is always the result of blowups and logarithmic transforms.

Rational blowdown changes the topology of the manifold X; while χ_h remains the same, c is decreased by $p-3$. So, an obvious problem would be

Problem 12. *Are any two homeomorphic simply-connected smooth 4-manifolds related via a sequence of generalized logarithmic transforms on tori?*

As already pointed out, there are two cases.

(1) T is essential in homology
(2) T is null-homologous

This leads to:

Problem 13. *Can a generalized logarithmic transform on a homologicaly essential torus be obtained via a sequence of generalized logarithmic transforms on null-homologous tori?*

For the rest of the lecture we will discuss these last two problems.

5. Cobordisms between 4-manifolds

.

Let X_1 and X_2 be two homeomorphic simply-connected smooth 4-manifolds. Early results of C.T.C. Wall show that there is an h–cobordism W^5 between X_1 and X_2 obtained from $X_1 \times I$ by attaching n $2-$handles and n $3-$handles. A long standing problem that still remains open is:

Problem 14. *Can W^5 can be chosen so that $n = 1$.*

Let's explore the consequences if we can assume $n = 1$. We can then describe the $h-$cobordism W^5 as follows. First, let W_1 be the cobordism from X_1 to $X_1 \# S^2 \times S^2$ given by attaching the $2-$handle to X_1. To complete W^5 we then would add the $3-$handle. Dually, this is equivalent to attaching a $2-$handle to X_2. So let W_2 be the cobordism from X_2 to $X_2 \# S^2 \times S^2$ given by attaching this $2-$handle to X_2. Then $W^5 = W_1 \cup_f (-W_2)$ for a suitable diffeomorphism $f\colon X_1 \# S^2 \times S^2 \to X_2 \# S^2 \times S^2$. Let A be any of the standard spheres in $S^2 \times S^2$.

Then the complexity of the h-cobordism can be measured by the type k, which is half the minimum of the number of intersection points between A and $f(A)$ (as $A \cdot f(A) = 0$ there are k positive intersection points and k negative intersection points). This complexity has been studied in [19]. A key observation is that if $k = 1$, then a neighborhood of $A \cup f(A)$ is diffeomorphic to an embedding of twin spheres in S^4 and that its boundary is the three-torus T^3. A further observation is that X_2 is then obtained from X_1 by removing a neighborhood of a null-homologous torus T embedded in X_1 (with trivial normal bundle) and sewing it back in differently. Thus when $k = 1$, X_2 is obtained from X_1 by a generalized generalized logarithmic transform on a *null-homologous* torus.

This points out that the answers to Problems 12 and 13 are related to the complexity k of h-cobordisms. We expect that the answer to Problem 13 is NO and that ordinary generalized logarithmic transforms on homologically essential tori will provide examples of homeomorphic X_1 and X_2 that require h–cobordisms with arbitrarily large complexity.

Independent of this, an important next step is to study complexity $k > 1$ h-cobordisms. Here, new surgical techniques are suggested. In particular, the neighborhood of $A \cup f(A)$ above is diffeomorphic to the neighborhood N' of two $2-$spheres embedded in S^4 with $2k$ points of intersection. Let N be obtained from N' with one of the $2-$spheres surgered out. Then it can be shown that X_1 is obtained from X_2 by removing an embedding of N' and regluing along a diffeomorphism of its boundary. This could lead to a useful generalization of logarithmic transforms along null-homologous tori. It would then be important to compute its effect on the Seiberg-Witten invariants, and reinterpret generalized logarithmic transforms from this point of view.

Round handlebody cobordisms. Suppose that X_1 and X_2 are two manifolds with the same c and χ_h. It follows from early work of Asimov [1] that there is a round handlebody cobordism W between X_1 and X_2. Thus X_1 can be obtained from X_2 by attaching a sequence of round $1-$handles and round $2-$handles. A round handle is just S^1 times a handle in one lower dimension. So for us a round $r-$handle is a copy of $S^1 \times (D^r \times D^{4-r})$ attached along $S^1 \times (S^{r-1} \times D^{4-r})$ (see [1] for definitions).

Problem 15. *Can W be chosen so that there are no round $1-$handles?*

For a moment, suppose that the answer to Problem 15 is Yes. Then W would consist of only round $2-$handles. It then follows that X_2 would be obtained from X_1 via a sequence of generalized logarithmic transforms on tori. Thus the answer to Problem 15 is tightly related to Problem 12.

Note that if X_1 and X_2 are round handlebody cobordant, then the only invariant preventing them from being homeomorphic is whether $t(X_1) = t(X_2)$. So suppose $t(X_1) = 0$ and $t(X_2) = 1$. If the answer to Problem 15 were yes, then one could change the second Stiefel-Whitney class via a sequence of generalized logarithmic transforms on tori. By necessity these tori cannot be null-homologous. So understanding new surgical operations that will change t without changing c, χ_h, and preserving the Seiberg-Witten invariants should provide new insights.

Problem 16. *Suppose two simply-connected smooth 4-manifolds have the same c, χ_h, number of Seiberg-Witten basic classes, and different t. Determine surgical operations that will transform one to the other.*

There are explicit examples of this phenomena amongst complex surfaces, e.g. two Horikawa surfaces with the same c and χ_h, but different t.

6. Modifying symplectic 4-manifolds

To finish up this lecture, we point out that all known constructions of (simply-connected) non-symplectic 4-manifolds can be obtained from symplectic 4-manifolds by performing logarithmic transforms on null-homologous Lagrangian tori with non-vanishing framing defect (cf. [10]). Let's look at a specific example of this phenomena. In particular, let's consider $E(n)_K$.

The elliptic surface $E(n)$ is the double branched cover of $S^2 \times S^2$ with branch set equal to four disjoint copies of $S^2 \times \{\text{pt}\}$ together with $2n$ disjoint copies of $\{\text{pt}\} \times S^2$. The resultant branched cover has $8n$ singular points (corresponding to the double points in the branch set), whose neighborhoods are cones on $\mathbf{RP}^3$. These are desingularized in the usual way, replacing their neighborhoods with cotangent bundles of S^2. The result is $E(n)$. The horizontal and vertical fibrations of $S^2 \times S^2$ pull back to give fibrations of $E(n)$ over $\mathbf{CP}^1$. A generic fiber of the vertical fibration is the double cover of S^2, branched over 4 points — a torus. This describes an elliptic fibration of $E(n)$. The generic fiber of the horizontal fibration is the double cover of S^2, branched over $2n$ points, and this gives a genus $n-1$ fibration on $E(n)$. This genus $n-1$ fibration has four singular fibers which are the preimages of the four $S^2 \times \{\text{pt}\}$'s in the branch set together with the spheres of self-intersection -2 arising from desingularization. The generic fiber T of the elliptic fibration meets a generic fiber Σ_{n-1} of the horizontal fibration in two points, $\Sigma_{n-1} \cdot T = 2$.

Now let K be a fibered knot of genus g, and fix a generic elliptic fiber T_0 of $E(n)$. Then in the knot surgery manifold

$$E(n)_K = (E(n) \setminus (T_0 \times D^2)) \cup (S^1 \times (S^3 \setminus N(K)),$$

each normal 2-disk to T_0 is replaced by a fiber of the fibration of $S^3 \setminus N(K)$ over S^1. Since T_0 intersects each generic horizontal fiber twice, we obtain a 'horizontal' fibration

$$h : E(n)_K \to \mathbf{CP}^1$$

of genus $2g + n - 1$.

This fibration also has four singular fibers arising from the four copies of $S^2 \times \{\text{pt}\}$ in the branch set of the double cover of $S^2 \times S^2$. Each of these gets blown up at $2n$ points in $E(n)$, and the singular fibers each consist of a genus g surface Σ_g of self-intersection $-n$ and multiplicity 2 with $2n$ disjoint 2-spheres of self-intersection -2, each meeting Σ_g transversely in one point. The monodromy around each singular fiber is (conjugate to) the diffeomorphism of Σ_{2g+n-1} which is the deck transformation η of the double cover of Σ_g, branched over $2n$ points. Another way to describe η is to take the hyperelliptic involution ω of Σ_{n-1} and to connect sum copies of Σ_g at the two points of a nontrivial orbit of ω. Then ω extends to the involution η of Σ_{2g+n-1}.

The fibration which we have described is not Lefschetz since the singularities are not simple nodes. However, it can be perturbed locally to be Lefschetz.

So in summary, if K is a fibered knot whose fiber has genus g, then $E(n)_K$ admits a locally holomorphic fibration (over $\mathbf{CP}^1$) of genus $2g + n - 1$ which has exactly four singular fibers. Furthermore, this fibration can be deformed locally to be Lefschetz.

There is another way to view these constructions. Consider the branched double cover of $\Sigma_g \times S^2$ whose branch set consists of two disjoint copies of $\Sigma_g \times \{\mathrm{pt}\}$ and $2n$ disjoint copies of $\{\mathrm{pt}\} \times S^2$. After desingularizing as above, one obtains a complex surface denoted $M(n,g)$. Once again, this manifold carries a pair of fibrations. There is a genus $2g+n-1$ fibration over S^2 and an S^2 fibration over Σ_g.

Consider first the S^2 fibration. This has $2n$ singular fibers, each of which consists of a smooth 2-sphere E_i, $i = 1, \ldots, 2n$, of self-intersection -1 and multiplicity 2, together with a pair of disjoint spheres of self-intersection -2, each intersecting E_i once transversely. If we blow down E_i we obtain again an S^2 fibration over Σ_g, but the ith singular fiber now consists of a pair of 2-spheres of self-intersection -1 meeting once, transversely. Blowing down one of these gives another S^2 fibration over Σ_g, with one less singular fiber. Thus blowing down $M(n,g)$ $4n$ times results in a manifold which is an S^2 bundle over Σ_g. This shows that (if $n > 0$) $M(n,g)$ is diffeomorphic to $(S^2 \times \Sigma_g)\#4n\,\overline{\mathbf{CP}}^2$.

The genus $2g+n-1$ fibration on $M(n,g)$ has 2 singular fibers. As above, these fibers consist of a genus g surface Σ_g of self-intersection $-n$ and multiplicity 2 with $2n$ disjoint 2-spheres of self-intersection -2, each meeting Σ_g transversely in one point. The monodromy of the fibration around each of these fibers is the deck transformation of the double branched cover of Σ_g. This is just the map η described above.

Let φ be a diffeomorphism of $\Sigma_g \setminus D^2$ which is the identity on the boundary. For instance, φ could be the monodromy of a fibered knot of genus g. There is an induced diffeomorphism Φ of $\Sigma_{2g+n-1} = \Sigma_g\#\Sigma_{n-1}\#\Sigma_g$ which is given by φ on the first Σ_g summand and by the identity on the other summands. Consider the twisted fiber sum

$$M(n,g)\#_\Phi M(n,g) = \{M(n,g)\setminus(D^2\times\Sigma_{2g+n-1})\}\cup_{\mathrm{id}\times\Phi}\{M(n,g)\setminus(D^2\times\Sigma_{2g+n-1})\}$$

where fibered neighborhoods of generic fibers Σ_{2g+n-1} have been removed from the two copies of $M(n,g)$, and they have been glued by the diffeomorphism $\mathrm{id}\times\Phi$ of $S^1 \times \Sigma_{2g+n-1}$.

In the case that φ is the monodromy of a fibered knot K, it can be shown that $M(n,g)\#_\Phi M(n,g)$ is the manifold $E(n)_K$ with the genus $2g+n-1$ fibration described above. To see this, we view S^2 as the base of the horizontal fibration. Then it suffices to check that the total monodromy map $\pi_1(S^2\setminus 4\,\mathrm{points}) \to \mathrm{Diff}(\Sigma_{2g+n-1})$ is the same for each. It is not difficult to see that if we write the generators of $\pi_1(S^2 \setminus 4\,\mathrm{points})$ as α, β, γ with α and β representing loops around the singular points of, say, the image of the first copy of $M(n,g)$ and basepoint in this image, and γ a loop around a singular point in the image of the second $M(n,g)$ then the monodromy map μ satisfies $\mu(\alpha) = \eta$, $\mu(\beta) = \eta$ and $\mu(\gamma)$ is $\varphi \oplus \omega \oplus \varphi^{-1}$, expressed as a diffeomorphism of $\Sigma_g\#\Sigma_{n-1}\#\Sigma_g$. That this is also the monodromy of $E(n)_K$ follows directly from its construction.

Now let $E(n)_g$ denote $E(n)$ fiber summed with $T^2 \times \Sigma_g$ along an elliptic fiber. The penultimate observation is that $E(n)_K$, viewed as $M(n,g)\#_\Phi M(n,g)$, is then the result of a sequence of generalized logarithmic transforms on null-homologous Lagrangian tori in $E(n)_g$. The effect of these surgeries is to change the monodromy of the genus $n+2g-1$ Lefschetz fibration (over $\mathbf{CP}^1$) on $E(n)_g$. This is accomplished by doing a $1/n$, with respect to the natural Lagrangian framing, generalized logarithmic transform on these Lagrangian tori (cf. [9,10]). The final observation is

that if the Lagrangian framing of these tori differs from the null-homologous framing (cf. [10]), then a $1/n$ log transformations on T with respect to the null-homologous framing can be shown, by computing Seiberg-Witten invariants, to result in non-symplectic 4-manifolds. Careful choices of these tori and framings will result in manifolds homotopy equivalent to $M(n,g)\#_\Phi M(n,g)$ (cf. [9]).

References

[1] Daniel Asimov, *Round handles and homotopy of nonsingular vector fields*, Bull. Amer. Math. Soc. **81** (1975), 417–419. MR0370658 (51 #6884)

[2] Stefan Bauer and Mikio Furuta, *A stable cohomotopy refinement of Seiberg-Witten invariants. I*, Invent. Math. **155** (2004), 1–19. MR2025298

[3] S. K. Donaldson, *An application of gauge theory to four-dimensional topology*, J. Differential Geom. **18** (1983), 279–315. MR710056 (85c:57015)

[4] Ronald J. Fintushel, Jongil Park, and Ronald J. Stern, *Rational surfaces and symplectic 4-manifolds with one basic class*, Algebr. Geom. Topol. **2** (2002), 391–402 (electronic). MR1917059 (2003f:57063)

[5] Ronald Fintushel and Ronald J. Stern, *Immersed spheres in 4-manifolds and the immersed Thom conjecture*, Turkish J. Math. **19** (1995), 145–157. MR1349567 (96j:57036)

[6] ———, *Rational blowdowns of smooth 4-manifolds*, J. Differential Geom. **46** (1997), 181–235. MR1484044 (98j:57047)

[7] ———, *Knots, links, and 4-manifolds*, Invent. Math. **134** (1998), 363–400. MR1650308 (99j:57033)

[8] ———, *The canonical class of a symplectic 4-manifold*, Turkish J. Math. **25** (2001), 137–145. MR1829084 (2002g:57053)

[9] ———, *Families of simply connected 4-manifolds with the same Seiberg-Witten invariants*, Topology **43** (2004), 1449–1467. MR2081432

[10] ———, *Invariants for Lagrangian Tori*, Geometry and Topology **8** (2004), 949–968.

[11] ———, *Double node neighborhoods and families of simply connected 4-manifolds with $b^+ = 1$* (2004), http://front.math.ucdavis.edu/math.GT/0412126 (preprint).

[12] Michael Hartley Freedman, *The topology of four-dimensional manifolds*, J. Differential Geom. **17** (1982), 357–453. MR679066 (84b:57006)

[13] Robert E. Gompf and András I. Stipsicz, *4-manifolds and Kirby calculus*, Graduate Studies in Mathematics, vol. 20, American Mathematical Society, Providence, RI, 1999, ISBN 0-8218-0994-6. MR1707327 (2000h:57038)

[14] Dieter Kotschick, *On manifolds homeomorphic to* $\mathbf{C}\mathrm{P}^2\#8\overline{\mathbf{C}\mathrm{P}}^2$, Invent. Math. **95** (1989), 591–600. MR979367 (90a:57047)

[15] P. B. Kronheimer and T. S. Mrowka, *The genus of embedded surfaces in the projective plane*, Math. Res. Lett. **1** (1994), 797–808. MR1306022 (96a:57073)

[16] T. J. Li and A. Liu, *Symplectic structure on ruled surfaces and a generalized adjunction formula*, Math. Res. Lett. **2** (1995), 453–471. MR1355707 (96m:57052)

[17] Marcos Mariño, Gregory Moore, and Grigor Peradze, *Superconformal invariance and the geography of four-manifolds*, Comm. Math. Phys. **205** (1999), 691–735. MR1711332 (2001h:57037)

[18] John W. Morgan, Tomasz S. Mrowka, and Zoltán Szabó, *Product formulas along T^3 for Seiberg-Witten invariants*, Math. Res. Lett. **4** (1997), 915–929. MR1492130 (99f:57039)

[19] John W. Morgan and Zoltán Szabó, *On h-cobordisms and Seiberg-Witten invariants*, Topics in Symplectic 4-Manifolds (Irvine, CA, 1996), First Int. Press Lect. Ser., I, Internat. Press, Cambridge, MA, 1998, pp. 117–124. MR1635699 (99h:57061)

[20] Peter Ozsváth and Zoltán Szabó, *On Park's exotic smooth four-manifolds* (2004), http://front.math.ucdavis.edu/math.GT/0411218 (preprint).

[21] Jongil Park, *The geography of Spin symplectic 4-manifolds*, Math. Z. **240** (2002), 405–421. MR1900318 (2003c:57030)

[22] ———, *Simply connected symplectic 4-manifolds with $b_2^+ = 1$ and $c_1^2 = 2$* (2004), http://front.math.ucdavis.edu/math.GT/0311395 (to appear in Invent. Math.)

[23] Jongil Park, Andras Stipsicz, and Zoltán Szabó, *Exotic smooth structures on* $\mathbf{CP}^2\#5\overline{\mathbf{CP}}^2$ (2004), http://front.math.ucdavis.edu/math.GT/0412216 (preprint).

[24] Ulf Persson, Chris Peters, and Gang Xiao, *Geography of spin surfaces*, Topology **35** (1996), 845–862. MR1404912 (98h:14046)

[25] Andras Stipsicz and Zoltán Szabó, *An exotic smooth structure on* $\mathbf{CP}^2\#6\overline{\mathbf{CP}}^2$ (2004), http://front.math.ucdavis.edu/math.GT/0411258 (preprint).

[26] Margaret Symington, *Symplectic rational blowdowns*, J. Differential Geom. **50** (1998), 505–518. MR1690738 (2000e:57043)

[27] Clifford Henry Taubes, *The Seiberg-Witten and Gromov invariants*, Math. Res. Lett. **2** (1995), 221–238. MR1324704 (96a:57076)

[28] ———, *Seiberg-Witten and Gromov invariants for symplectic* 4*-manifolds*, First International Press Lecture Series, vol. 2, International Press, Somerville, MA, 2000, ISBN 1-57146-061-6, Edited by Richard Wentworth. MR1798809 (2002j:53115)

[29] ———, *The Seiberg-Witten invariants and 4-manifolds with essential tori*, Geom. Topol. **5** (2001), 441–519 (electronic). MR1833751 (2002d:57025)

[30] Edward Witten, *Monopoles and four-manifolds*, Math. Res. Lett. **1** (1994), 769–796. MR1306021 (96d:57035)

Department of Mathematics, University of California, Irvine, CA 92697
E-mail address: rstern@uci.edu