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Surgery on nullhomologous tori and simply connected 4-manifolds with $b^+ = 1$

Ronald Fintushel and Ronald J. Stern

Abstract

For $5 \leqslant k \leqslant 8$, we show that an infinite family of exotic smooth structures on $\mathbf{CP}^2 \# k\, \overline{\mathbf{CP}}^2$ can be obtained by $1/n$-surgeries on a single embedded nullhomologous torus in a manifold R_k which is homeomorphic to $\mathbf{CP}^2 \# k\, \overline{\mathbf{CP}}^2$.

1. *Introduction*

In the past few years, there has been significant progress on the problem of finding exotic smooth structures on the manifolds $P_m = \mathbf{CP}^2 \# m\, \overline{\mathbf{CP}}^2$. The initial step was taken by Park [**16**], who found the first exotic smooth structure on P_7, and whose ideas renewed the interest in this subject. Ózsvath and Szabó proved that Park's manifold is minimal [**15**], and Stipsicz and Szabó used a technique similar to that of Park's to construct an exotic structure on P_6 [**19**]. Shortly thereafter, the authors of this paper developed a technique for producing infinite families of smooth structures on P_m, $6 \leqslant m \leqslant 8$ (see [**8**]), and Park, Stipsicz, and Szabó showed that this can be applied to the case $m = 5$ (see [**8**, **18**]).

It is the goal of this paper to better understand the underlying mechanism which produces infinitely many distinct smooth structures on P_m, $5 \leqslant m \leqslant 8$. As we explain below, all these constructions start with the elliptic surface $E(1) = P_9$: perform a knot surgery using a family of twist knots indexed by an integer n [**7**], then blow the result up several times in order to find a suitable configuration of spheres that can be rationally blown down [**5**] to obtain a smooth structure on P_m that is distinguished by the integer n. We shall explain how this can be accomplished by surgery on nullhomologous tori in a manifold R_m homeomorphic to P_m, $5 \leqslant m \leqslant 8$. In other words, we shall find a nullhomologous torus Λ_m in R_m so that $1/n$-surgery on Λ_m preserves the homeomorphism type of R_m, but changes the smooth structure of R_m in a way that depends on n. Presumably, R_m is diffeomorphic to P_m, but we have not yet been able to show this in general. Our hope is that by better understanding Λ_m and its properties, one will be able to find similar nullhomologous tori in P_m, for $m < 5$.

2. *A short history of simply connected* 4*-manifolds with* $b^+ = 1$

It is a basic problem of 4-manifold topology to understand the smooth structures on the complex projective plane $\mathbf{CP}^2$. Thus, one is interested in knowing the smallest m for which $P_m = \mathbf{CP}^2 \# m\, \overline{\mathbf{CP}}^2$ admits an exotic smooth structure. The first such example was produced by Donaldson in the historic paper [**2**], where it was shown that the Dolgachev surface $E(1)_{2,3}$, the result of performing log transforms of orders 2 and 3 on the rational elliptic surface $E(1) = P_9$ [**3**], is homeomorphic but not diffeomorphic to P_9. This breakthrough example provided the first known instance of an exotic smooth structure on a simply connected 4-manifold.

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Later, work of Friedman and Morgan [9] showed that the integers p and q, for $p, q > 1$, are smooth invariants. The fact that $E(1)_{p,q}$ is not diffeomorphic to $E(1)_{p',q'}$ for $\{p, q\} \neq \{p', q'\}$, $p, q, p', q' > 1$ persists even after an arbitrary number of blowups [4]; however, no minimal exotic smooth structures are currently known for P_m where $m \geqslant 10$.

In the late 1980s, Kotschick [11] proved that the Barlow surface, known to be homeomorphic to P_8, is not diffeomorphic to it. However, in following years, the subject of simply connected smooth 4-manifolds with $b^+ = 1$ languished because of lack of suitable examples. As we mentioned above, largely due to the example of Park [16] of an exotic smooth structure on P_7, this topic has again become active.

Here is an outline of a version of Park's example: consider $E(1)$ with an elliptic fibration whose singular fibers are four nodal fibers and an I_8-fiber. (An I_n-fiber is comprised of a circular plumbing of n 2-spheres of self-intersection -2; see [1].) This elliptic fibration has a section which is an exceptional curve E (of self-intersection -1). Blow up $E(1)$ four times, at the double points of the four nodal fibers. Then in $E(1)\#4\overline{\mathbf{CP}}^2 \cong P_{13}$, we find a configuration of 2-spheres consisting of E, four disjoint spheres of self-intersection -4, each intersecting E once, and the I_8-fiber, which intersects E in exactly one 2-sphere, and the I_8-fiber is disjoint from the 4-spheres of self-intersection -4.

The transverse intersections of E with the 4-spheres of self-intersection -4 can be smoothed to obtain a 2-sphere of self-intersection -9. Together with spheres from the I_8-fiber, we obtain a linear configuration of 2-spheres:

$$\overset{-9}{\bullet}\!-\!\!\!-\!\overset{-2}{\bullet}\!-\!\!\!-\!\overset{-2}{\bullet}\!-\!\!\!-\!\overset{-2}{\bullet}\!-\!\!\!-\!\overset{-2}{\bullet}\!-\!\!\!-\!\overset{-2}{\bullet}$$

A regular neighborhood of this configuration has as its boundary the lens space $L(49, -6)$, and as we explain below, this lens space bounds a rational homology ball. This means that this configuration can be rationally blown down [5], reducing b^- by 6. One obtains a manifold P with $b^+ = 1$ and $b^- = 7$. It is not difficult to show that P is simply connected, and so it is homeomorphic to P_7. It follows from Seiberg–Witten theory that P is not diffeomorphic to P_7. This will be explained below. While this is not precisely the description of the manifold that was given in [16] (and it is not even clear that P is diffeomorphic to the example of [16]), the construction given here is similar to that of Park.

Stipsicz and Szabó improved on Park's example by finding a more complicated configuration in a larger blowup of $E(1)$, yet one which could be rationally blown down to get an even smaller manifold, homeomorphic but not diffeomorphic to P_6.

For some time, even after these examples, it was suspected that P_m, for $m < 9$, would support only finitely many distinct smooth structures. This was due to the fact that until [8] the only technique available for producing infinitely many distinct smooth structures on a given smooth 4-manifold X was to require that X contain a minimal genus torus with trivial normal bundle and representing a nontrivial homology class. It is known that P_m, for $m < 9$, contain no such tori. Thus, it is the goal of this paper to better understand the techniques for producing infinitely many distinct smooth structures and to better understand the examples of [8].

3. *Seiberg–Witten invariants, rational blowdowns, and knot surgery*

3.1. *Seiberg–Witten invariants*

Let X be a simply connected oriented 4-manifold with $b^+_X = 1$, a given orientation of $H^2_+(X; \mathbf{R})$, and a given metric g. The Seiberg–Witten invariant depends on the metric g and a self-dual 2-form as follows. There is a unique g-self-dual harmonic 2-form $\omega_g \in H^2_+(X; \mathbf{R})$ with $\omega_g^2 = 1$ and corresponding to the positive orientation. (Often ω_g is called a *period point* for the metric g.) Fix a characteristic homology class $k \in H_2(X;\mathbf{Z})$. Given a pair (A,ψ), where A is a connection in the complex line bundle whose first Chern class is the Poincaré dual $\widehat{k} = \frac{i}{2\pi}[F_A]$ of k and ψ is a section of the bundle W^+ of self-dual spinors for the associated *spin*^{c} structure, the perturbed Seiberg–Witten equations are:

$$
\begin{aligned}
D_A\psi &= 0\\
F_A^+ &= q(\psi) + i\eta
\end{aligned}
$$

where F_A^+ is the self-dual part of the curvature F_A, D_A is the twisted Dirac operator, η is a self-dual 2-form on X, and q is a quadratic function. Write $\mathrm{SW}_{X,g,\eta}(k)$ for the corresponding invariant. As the pair (g,η) varies, $\mathrm{SW}_{X,g,\eta}(k)$ can change only at those pairs (g,η) for which there are solutions with $\psi = 0$. These solutions occur for pairs (g,η) satisfying $(2\pi\widehat{k}+\eta)\cdot\omega_g = 0$. This last equation defines a wall in $H^2(X;\mathbf{R})$.

The point ω_g determines a component of the double cone consisting of elements of $H^2(X;\mathbf{R})$ of positive square. We prefer to work with $H_2(X;\mathbf{R})$. The dual component is determined by the Poincaré dual H of ω_g. An element $H' \in H_2(X;\mathbf{R})$ of positive square lies in the same component as H if $H'\cdot H > 0$. If $(2\pi\widehat{k}+\eta)\cdot\omega_g \neq 0$ for a generic η, $\mathrm{SW}_{X,g,\eta}(k)$ is well defined, and its value depends only on the sign of $(2\pi\widehat{k}+\eta)\cdot\omega_g$. Write $\mathrm{SW}^+_{X,H}(k)$ for $\mathrm{SW}_{X,g,\eta}(k)$ if $(2\pi\widehat{k}+\eta)\cdot\omega_g > 0$, and $\mathrm{SW}^-_{X,H}(k)$ in the other case.

The invariant $\mathrm{SW}_{X,H}(k)$ is defined by $\mathrm{SW}_{X,H}(k) = \mathrm{SW}^+_{X,H}(k)$ if $(2\pi\widehat{k})\cdot\omega_g > 0$, or dually, if $k\cdot H > 0$, and $\mathrm{SW}_{X,H}(k) = \mathrm{SW}^-_{X,H}(k)$ if $k\cdot H < 0$. The wall-crossing formula [**12**, **13**] states that if H', H'' are elements of positive square in $H_2(X;\mathbf{R})$ with $H'\cdot H > 0$ and $H''\cdot H > 0$, then if $k\cdot H' < 0$ and $k\cdot H'' > 0$,

$$
\mathrm{SW}_{X,H''}(k) - \mathrm{SW}_{X,H'}(k) = (-1)^{1+\frac{1}{2}d(k)}
$$

where $d(k) = \frac{1}{4}(k^2 - (3\,\mathrm{sign} + 2\,\mathrm{e})(X))$ is the formal dimension of the Seiberg–Witten moduli spaces.

Furthermore, in case $b^- \leqslant 9$, the wall-crossing formula, together with the fact that $\mathrm{SW}_{X,H}(k) = 0$ if $d(k) < 0$, implies that $\mathrm{SW}_{X,H}(k) = \mathrm{SW}_{X,H'}(k)$ for any H' of positive square in $H_2(X;\mathbf{R})$ with $H\cdot H' > 0$. So in case $b^+_X = 1$ and $b^-_X \leqslant 9$, there is a well-defined Seiberg–Witten invariant, $\mathrm{SW}_X(k)$. If $\mathrm{SW}_X(k) \neq 0$, k is called a *basic class* of X.

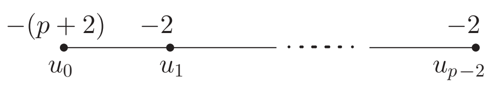
It is convenient to view the Seiberg–Witten invariant as an element of the integral group ring $\mathbf{Z}H_2(X)$. For $k \in H_2(X)$, we let t_k denote the corresponding element in $\mathbf{Z}H_2(X)$. Then, the Seiberg–Witten invariant of X is

$$
\mathcal{SW}_{X,H} = \sum \mathrm{SW}_{X,H}(k)\cdot t_k.
$$

An important property of the Seiberg–Witten invariant is that if X admits a metric g of positive scalar curvature, then for the Poincaré dual H of ω_g, we have $\mathcal{SW}_{X,H} = 0$. In particular, for $m \leqslant 9$, $\mathcal{SW}_{P_m} = 0$.

3.2. *Rational blowdowns*

Let C_p be the smooth 4-manifold obtained by plumbing $(p-1)$ disk bundles over the 2-sphere according to the diagram:



Then, the classes of the 0-sections have self-intersections $u_0^2 = -(p+2)$ and $u_i^2 = -2$, $i = 1,\ldots,p-2$. The boundary of C_p is the lens space $L(p^2, 1-p)$ which bounds a rational ball

B_p with $\pi_1(B_p) = \mathbf{Z}_p$ and $\pi_1(\partial B_p) \to \pi_1(B_p)$ surjective. If C_p is embedded in a 4-manifold X, then the rational blowdown manifold $X_{(p)}$ of [5] is obtained by replacing C_p with B_p, that is, $X_{(p)} = (X \smallsetminus C_p) \cup B_p$. This construction is independent of the choice of gluing map.

Rational Blowdown Theorem [5]. *Let X be a simply connected 4-manifold containing the configuration of 2-spheres, C_p. If $X \smallsetminus C_p$ is also simply connected, then so is the rational blowdown manifold $X_{(p)}$. The homology, $H_2(X_{(p)};\mathbf{R})$, may be identified with the orthogonal complement of the classes u_i, $i = 0, \dots, p-2$ in $H_2(X;\mathbf{R})$.*

Given a characteristic homology class $k \in H_2(X_{(p)};\mathbf{Z})$, there is a lift $\widetilde{k} \in H_2(X;\mathbf{Z})$ which is characteristic and for which the dimensions of moduli spaces agree, that is, $d_{X_{(p)}}(k) = d_X(\widetilde{k})$. If $b^+_X > 1$, then $SW_{X_{(p)}}(k) = SW_X(\widetilde{k})$. In case $b^+_X = 1$, if $H \in H_2^+(X;\mathbf{R})$ is orthogonal to all the u_i, then it can also be viewed as an element of $H_2^+(X_{(p)};\mathbf{R})$, and $SW_{X_{(p)},H}(k) = SW_{X,H}(\widetilde{k})$.

3.3. *Knot surgery*

Let X be a 4-manifold which contains a homologically essential torus T of self-intersection 0, and let K be a knot in S^3. Let $N(K)$ be a tubular neighborhood of K in S^3, and let $T \times D^2$ be a tubular neighborhood of T in X. Then the knot surgery manifold X_K is defined by:

$$X_K = (X \smallsetminus (T \times D^2)) \cup (S^1 \times (S^3 \smallsetminus N(K)).$$

The two pieces are glued together in such a way that the homology class $[\mathrm{pt} \times \partial D^2]$ is identified with $[\mathrm{pt} \times \lambda]$, where λ is the class of a longitude of K. For example, if X is a simply connected elliptic surface with a (spherical) section S of self-intersection n and one performs knot surgery on the fiber T of this fibration, then the gluing condition implies that in X_K there is a pseudosection S_K of genus equal to the genus of the knot K and with self-intersection n. By 'pseudosection' we mean that the intersection number $S_K \cdot F = 1$. However, X_K need no longer be an elliptic surface. This surface S_K is constructed by removing a disk from S where it intersects the fiber T and replacing this disk by a Seifert surface for the knot K.

One can also interpret X_K as a fiber sum. Let M_K denote the 3-manifold obtained from 0-framed surgery on K in S^3. Then

$$X_K = X \#_{T = S^1 \times m} S^1 \times M_K$$

where m is a meridian of K.

The gluing condition does not, in general, completely determine the diffeomorphism type of X_K; however, if we take X_K to be any manifold constructed in this fashion and if, for example, T has a cusp neighborhood, then the Seiberg–Witten invariant of X_K is completely determined by the Seiberg–Witten invariant of X and the symmetrized Alexander polynomial Δ^s of K.

Knot Surgery Theorem [7]. *Let X be a 4-manifold which contains a homologically essential torus T of self-intersection 0 whose H_1 is generated by vanishing cycles, and let K be a knot in S^3. The Seiberg–Witten invariant of the knot surgery manifold X_K is given by*

$$\mathcal{SW}_{X_K} = \mathcal{SW}_X \cdot \Delta^s(t^2)$$

where t represents the homology class of the torus T. Furthermore, if X and $X \smallsetminus T$ are simply connected, then so is X_K.

4. *Double node neighborhoods and knot surgery*

A simply connected elliptic surface is fibered over S^2 with a smooth fiber torus and with singular fibers. The most generic type of singular fiber is a nodal fiber (an immersed 2-sphere with one transverse positive double point). The monodromy of a nodal fiber is D_a, a Dehn twist

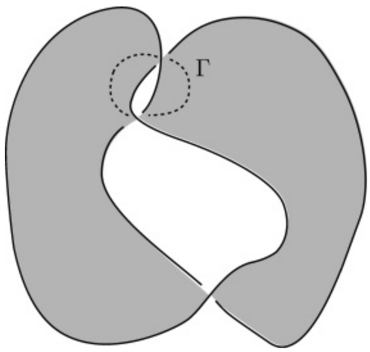


FIGURE 1. $K =$ *unknot.*

around the 'vanishing cycle' $a \in H_1(F;\mathbf{Z})$, where F is a smooth fiber of the elliptic fibration. The vanishing cycle a is represented by a nonseparating loop on the smooth fiber and the nodal fiber is obtained by collapsing this vanishing cycle to a point to create a transverse self-intersection. The vanishing cycle bounds a 'vanishing disk', a disk of relative self-intersection -1 with respect to the framing of its boundary given by pushing the loop off itself on the smooth fiber.

An I_2-fiber consists of a pair of 2-spheres of self-intersection -2, which are plumbed at two points. The monodromy of an I_2-fiber is D_a^2, which is also the monodromy of a pair of nodal fibers with the same vanishing cycle. This means that an elliptic fibration, which contains an I_2-fiber, can be perturbed to contain two nodal fibers with the same vanishing cycle.

A *double node neighborhood* D is a fibered neighborhood of an elliptic fibration, which contains exactly two nodal fibers with the same vanishing cycle. If F is a smooth fiber of D, there is a vanishing class a that bounds vanishing disks in the two different nodal fibers, and these give rise to a sphere V of self-intersection -2 in D.

In [**8**] we showed how performing knot surgery in a double node neighborhood D in $E(1)$ can give rise to an immersed pseudosection of self-intersection -1 in $E(1)_K$. Let us review a version of this construction. Consider the knot K of Figure 1. Of course, this is just the unknot, and we see a Seifert surface Σ of genus one. Let Γ be the loop which runs through both half-twists in the clasp. Then Γ satisfies the two key conditions of [**8**]:

(i) Γ bounds a disk in S^3 which intersects K at exactly two points.
(ii) The linking number in S^3 of Γ with its pushoff on Σ is $+1$.

It follows from these properties that Γ bounds a punctured torus in $S^3 \smallsetminus K$.

It is known that $E(1)$ admits an elliptic fibration with two nodal fibers: an I_2-fiber and an I_8-fiber [**17**]. As above, this fibration can be perturbed so that the I_2-fiber gives us a double node neighborhood D with vanishing cycle a. Consider the result of knot surgery in D using the knot K and the fiber F of $E(1)$. In the knot surgery construction, one is free to make any choice of gluing as long as a longitude of K is sent to the boundary circle of a normal disk to F. We choose the gluing so that the class of a meridian m of K is sent to the class of $a \times \{pt\}$ in $H_1(\partial(D \smallsetminus N(F));\mathbf{Z}) = H_1(F \times \partial D^2;\mathbf{Z})$. Note that the result of knot surgery

$$E(1)_K = E(1) \smallsetminus N(F) \cup S^1 \times (S^3 \smallsetminus N(K)) = E(1) \smallsetminus N(F) \cup T^2 \times D^2$$

because K is the unknot. Since any diffeomorphism of $\partial(E(1) \smallsetminus N(F))$ extends over all of $E(1) \smallsetminus N(F)$, we see that $E(1)_K$ is diffeomorphic to $E(1)$.

There is a genus one pseudosection S_K in $E(1)_K$ which is formed using the genus one Seifert surface Σ. The self-intersection of S_K is -1. The loop Γ sits on S_K, and by condition (i) it bounds a twice-punctured disk Δ in $\{pt\} \times \partial(S^3 \smallsetminus N(K))$ where $\partial\Delta = \Gamma \cup m_1 \cup m_2$ where the m_i are the meridians of K. The meridians m_i bound disjoint vanishing disks Δ_i in $D \smallsetminus N(F)$

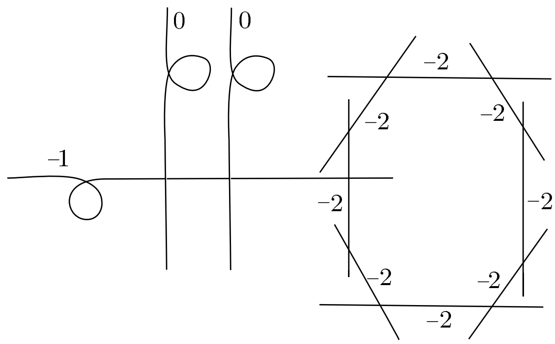


FIGURE 2.

since they are identified with disjoint loops each of which represents the class of $a \times \{pt\}$ in $H_1(\partial(D \smallsetminus N(F)); \mathbf{Z})$. (Our use of the terminology 'vanishing disk' is not entirely standard. Sometimes these disks are referred to as 'Lefschetz thimbles'.) Hence in D_K, the result of knot surgery on D, the loop $\Gamma \subset S_K$ bounds a disk $U = \Delta \cup \Delta_1 \cup \Delta_2$. By construction, the relative self-intersection of U relative to the framing given by the pushoff of Γ in S_K is $+1 - 1 - 1 = -1$. (This uses condition (ii).) Furthermore, $U \cap S_K = \Gamma$. This means that the relative normal bundle of U has Euler number -1; hence, it has a section which intersects U in a single point.

Since Γ is nonseparating in S_K, surgery on it kills $\pi_1(S_K)$. Ambient surgery may be performed in D_K by removing an annular neighborhood of Γ and replacing it with a pair of disks U', U'' as obtained above. These disks intersect in a single point, and this is precisely the complex-algebraic model of a nodal intersection. This means that we can represent the homology class of the pseudosection $[S_K]$ in $H_2(E(1)_K; \mathbf{Z})$ by an immersed sphere S' with one positive double point.

With these as preliminaries, our goal for the remainder of this paper is to construct for every $5 \leqslant m \leqslant 8$, a manifold R_m that is homeomorphic to P_m, has vanishing Seiberg–Witten invariants, and contains a nullhomologous torus Λ_m with the property that a $1/n$-surgery on Λ_m (with respect to the nullhomologous framing) yields a smooth structure on P_m distinguished by the integer n. We conjecture that R_m is diffeomorphic to P_m, but we are unable to show this at this time. We start with the $m = 8$ case in the next section.

5. *Infinite families homeomorphic to* $\mathbf{CP}^2 \# 8\, \overline{\mathbf{CP}}^2$

Our goal is to construct a manifold R_8, homeomorphic to P_8, with trivial Seiberg–Witten invariant and containing an embedded nullhomologous torus T such that $1/n$-surgery on T yields a smooth structure on P_8 which is distinguished by the integer n.

The construction of the previous section shows that when K is the unknot, then in $E(1) \cong E(1)_K$ one has the configuration consisting of the immersed 2-sphere S' with a pair of disjoint nodal fibers, each intersecting S' once transversely. Also, S' intersects the I_8-fiber transversely at one point. This is illustrated in Figure 2.

At this stage there are three possibilities:

(1) Blow up the double point of S'. Then, in P_{10} we obtain a configuration consisting of the total transform S'' of S', which is a sphere of self-intersection -5, and the sphere of self-intersection -2 at which S' intersects I_8 (see Figure 3). This is the configuration C_3 which can be rationally blown down to obtain a manifold R_8 with $b^+ = 1$ and $b^- = 8$. It is easy to see that R_8 is simply connected; so R_8 is homeomorphic to P_8.

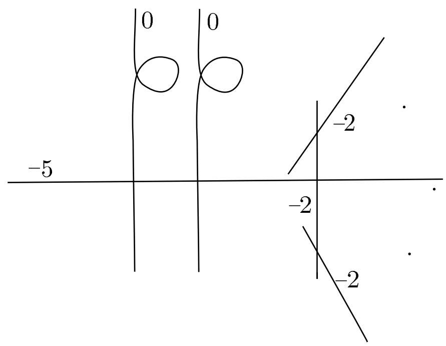


FIGURE 3.

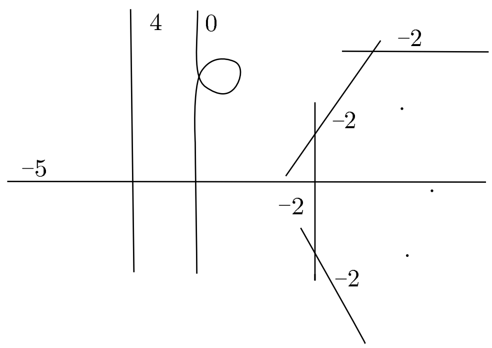


FIGURE 4.

(2) Blow up at the double point of S' as well as at the double point of one of the nodal fibers. Then, in P_{11} we get a configuration of 2-spheres consisting of S'', a transverse sphere F' of self-intersection -4, and three spheres from the I_8-fiber (see Figure 4). Smoothing the intersection of S'' and F' gives a sphere of self-intersection -7 and we obtain the configuration C_5 in P_{11}. Rationally blowing down C_5 gives a manifold R_7 homeomorphic to P_7.

(3) Blow up at the double point of S' as well as at the double points of both nodal fibers. Then, in P_{12} we get a configuration of 2-spheres consisting of S'', two disjoint transverse spheres F', F'' of self-intersection -4, and five spheres from the I_8-fiber (see Figure 5). Smoothing the intersections of S'', F' and F'' gives a sphere of self-intersection -9 and we obtain the configuration C_7 in P_{12}. Rationally blowing down C_7 gives a manifold R_6 homeomorphic to P_6.

We shall work with the the first case, and then indicate what needs to be done to take care of the other cases. In Case 1, we obtain a manifold $R = R_8$ which is homeomorphic to P_8. We conjecture that R is actually diffeomorphic to P_8, but for now it will suffice to see that it shares with P_8 the property that its Seiberg–Witten invariant vanishes.

PROPOSITION 5.1. $\mathcal{SW}_R = 0$.

Proof. We will show that R contains an embedded torus of self-intersection $+1$. Then the adjunction inequality will imply that $\mathcal{SW}_R = 0$. The upshot of using the unknot as K is that the loop v of Figure 6 bounds a disk of relative framing 0 with respect to the shaded genus 1 Seifert surface.

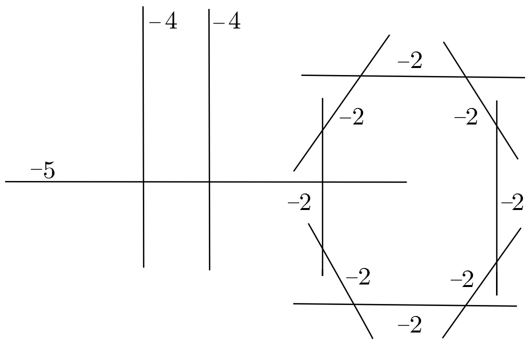


FIGURE 5.

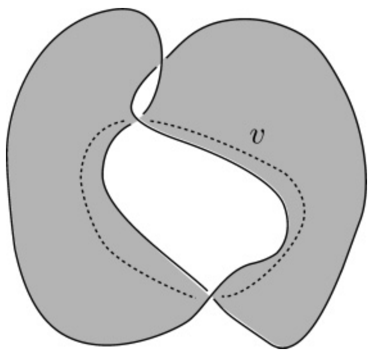


FIGURE 6.

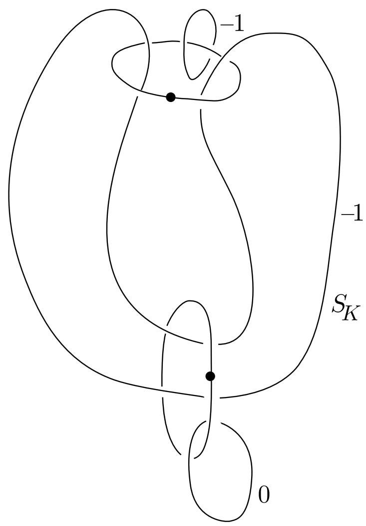


FIGURE 7.

It follows that a local picture of a neighborhood of the pseudosection S_K is given in Figure 7. The component labelled '0' is v, and the component on top labelled '-1' is Γ of Figure 1 arising from the double node construction.

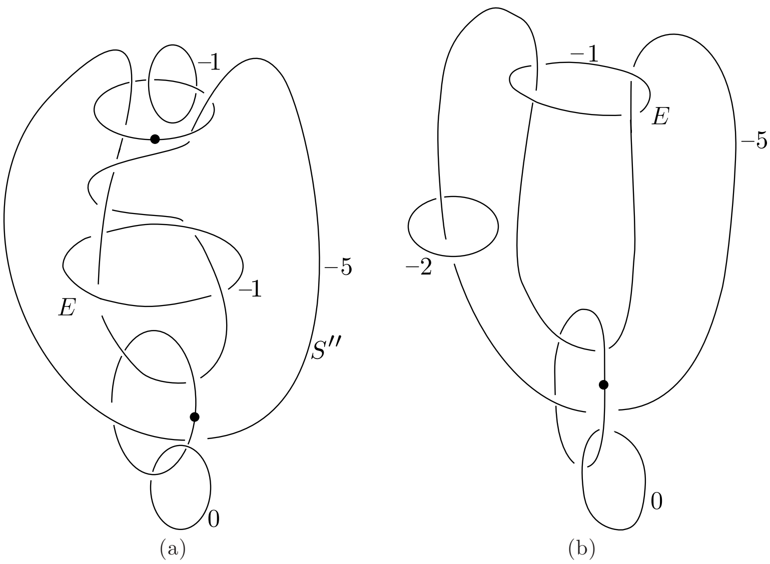


FIGURE 8.

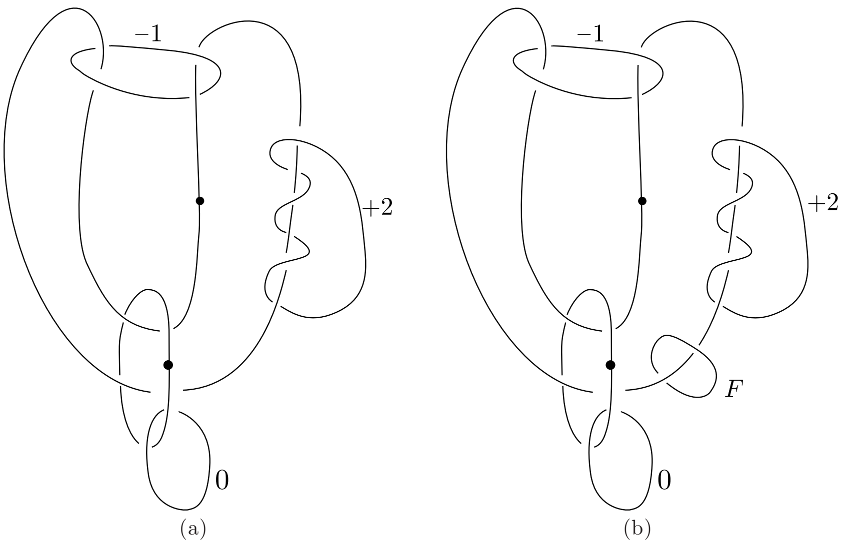


FIGURE 9.

Next, blow up to see $S'' = S - 2E$ (where E is the exceptional class) in Figure 8(a). In Figure 8(b), we cancel the 1-handle and the 2-handle labelled '-1'. The new 2-handle in this figure, labelled '-2', comes from the I_8-fiber as in Figure 3.

Rationally blow down the configuration $\{-5, -2\}$ to obtain Figure 9(a). The new loop labelled 'F' in Figure 9(b) is the intersection of a fiber of $E(1) = E(1)_K$ with the original section S and, therefore, with S''. This fiber can be taken to be far away from the region where the double node construction was performed.

Cancel the 1-handle and the 2-handle labelled '0' to obtain Figure 10(a), and then slide the handle labelled '+2' over the '-1' to obtain Figure 10(b).

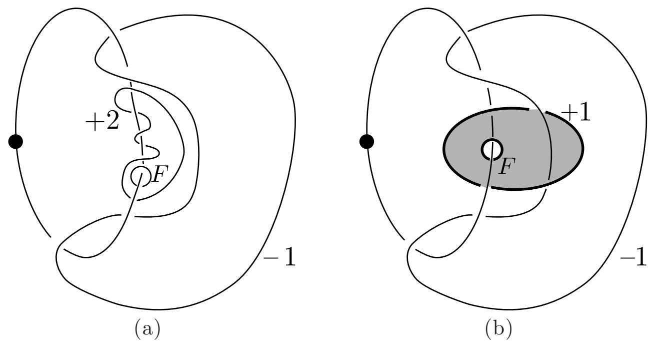


Figure 10.

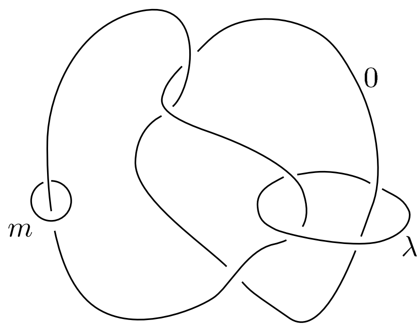


Figure 11.

Note that F bounds a punctured torus of square 0 outside of the neighborhood in Figure 10. Together with the shaded annulus in the figure and the +1-disk that the other boundary component of the annulus bounds, we get a torus of self-intersection +1 in R. As we have pointed out above, this implies that that $\mathcal{SW}_R = 0$. $\square$

Back in $E(1) = E(1)_K = E(1)\#_{F=S^1\times m} S^1 \times M_K$ there is a nullhomologous torus $\Lambda = S^1 \times \lambda$ where λ is the loop shown in Figure 11, which is a Kirby calculus depiction of $M_K = S^1 \times S^2$, since K is the unknot. Since Λ as well as the 3-manifold that it bounds ($S^1\times$ punctured torus) are disjoint from the regions where our constructions were made, Λ descends to a nullhomologous torus (which we still call Λ) in R. Let Q be the result of 0-surgery on $\Lambda \subset R$, where the '0-framing' is taken from the 0-framing on λ in Figure 11. After this surgery, the loop μ which bounds a normal disk to Λ, does not bound in Q. In fact, $H_2(Q)$ is the direct sum of $H_2(R)$ with a hyperbolic pair generated by Λ_0, the torus in Q corresponding to Λ, and a dual class represented by a torus built from the punctured torus that the longitude to λ (the surgery curve) bounds and the disk that the surgery curve bounds in Q. Thus $b^+(Q) = 2$.

Theorem 5.2. *The Seiberg–Witten invariant of Q is $\mathcal{SW}_Q = t^{-1} - t$, where $t = t_{S^1\times m} \in$ $\mathbf{Z}H_2(Q)$.*

Proof. The manifold Q is obtained by:

(1) double node surgery with $K =$ the unknot, blowing up, and then rationally blowing down,
(2) 0-surgery on Λ.

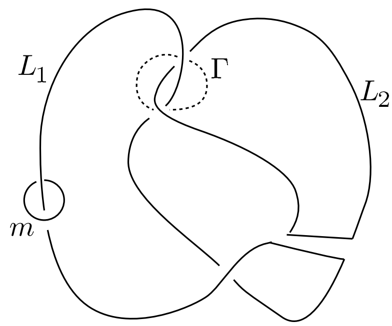


FIGURE 12.

Since Λ is disjoint from all the constructions in (1), the order in which (1) and (2) are performed is irrelevant. (Note that if we could exactly 'see' Λ embedded in P_8, step (1) would be unnecessary, and we could then use P_8 rather than R.)

In $E(1)_K \cong E(1)$, surgery is firstly done on Λ. Recall that $E(1)_K$ is a fiber sum

$$E(1)_K = E(1)\#_{F=S^1\times m} S^1 \times M_K$$

and M_K is the manifold given in Figure 11. The result of 0-surgery on Λ in $E(1)_K$ is the fiber sum

$$E(1)_{K,0} = E(1)\#_{F=S^1\times m} S^1 \times Y$$

where Y is the 3-manifold obtained from 0-surgery on λ in Figure 11.

We shall now need the sewn-up link exterior construction of Brakes and Hoste. We recall what this means. Let L be a link in S^3 with two *oriented* components L_1 and L_2. Fix tubular neighborhoods $N_i \cong S^1 \times D^2$ of L_i with $S^1 \times (\text{pt on } \partial D^2)$ a longitude of L_i, that is, nullhomologous in $S^3 \setminus L_i$. For any $A \in GL(2;\mathbf{Z})$ with $\det A = -1$, we the get a 3-manifold

$$s(L;A) = (S^3 \setminus \operatorname{int}(N_1 \cup N_2))/A$$

called a *sewn-up link exterior* by identifying ∂N_1 with ∂N_2 via a diffeomorphism inducing A in homology. For $n \in \mathbf{Z}$, let $A_n = \left(\begin{smallmatrix} -1 & 0 \\ -n & 1 \end{smallmatrix}\right)$. A simple calculation shows that $H_1(s(L;A_n);\mathbf{Z}) = \mathbf{Z} \oplus \mathbf{Z}_{n-2\ell}$ where ℓ is the linking number in S^3 of the two components L_1, L_2, of L. The second summand is generated by the meridian to either component.

We now refer to the proof of the Knot Surgery Theorem given in [**7**]. A key step in the proof (see [**7**, Figure 6]), which uses the work of Hoste [**10**], shows that $E(1)_{K,0}$ is diffeomorphic to

$$E(1)_L = E(1)\#_{F=S^1\times m} S^1 \times s(L, A_{-2})$$

where $L = L_1 \cup L_2$ is the link of Figure 12, that is, L is the Hopf link. The orientations on L_1 and L_2 are inherited from fixing an orientation on the knot K, for example, in Figure 11. Note that since the linking number of L_1 and L_2 is -1, we have $H_1(s(L;A_{-2});\mathbf{Z}) = \mathbf{Z} \oplus \mathbf{Z}$.

In $s(L, A_{-2})$ we see an embedded torus transverse to the meridian m obtained by sewing up a Seifert surface for L, and we also see a loop Γ which satisfies conditions (i) and (ii) for double node surgery.

The proof of the Knot Surgery Theorem shows that the Seiberg–Witten invariant of $E(1)_L$ can be calculated via skein moves (macarena). Note that we are now calculating the Seiberg–Witten invariant of a manifold with $b^+ = 2$. This calculation is shown in Figure 13. This figure depicts the fact that

$$\mathcal{SW}_{E(1)_L} = \mathcal{SW}_{E(1)_U} - (t - t^{-1})^2 \mathcal{SW}^-_{E(1)_{K_0}}$$

where U is the unlink and K_0 is the unknot (see [**7**, Section 3, equation (3)]).

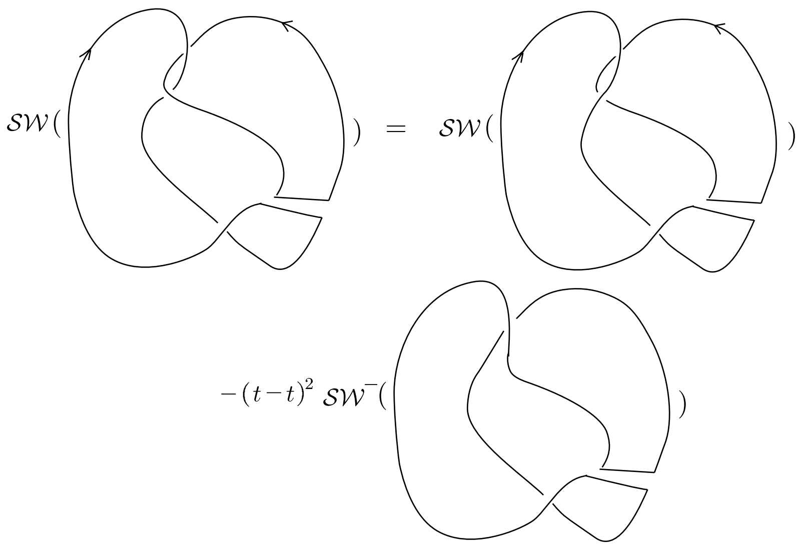


FIGURE 13.

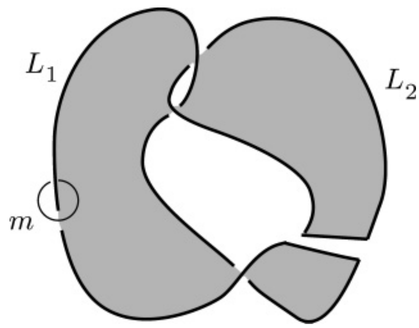


FIGURE 14.

A 2-sphere that separates the two components of the unlink gives rise to an essential 2-sphere in $E(1)_U$ of self-intersection 0. Thus $\mathcal{SW}_{E(1)_U}=0$. Since $E(1)_{K_0}=E(1)$, we have $\mathcal{SW}^-_{E(1)_{K_0}}=(t-t^{-1})^{-1}$. Thus, $\mathcal{SW}_{E(1)_L}=t^{-1}-t$. This completes our discussion of step (2).

Next, we carry out the constructions of step (1). In $E(1)_L$ there is a genus one pseudosection to which we can apply the double node construction. This pseudosection is the connected sum of a section in $E(1)$ with the torus in $\{\text{point}\}\times s(L, A_{-2})$ obtained by sewing up the shaded region in Figure 14. The necessary loop Γ is shown in Figure 12.

The result of the double node construction is an immersed genus 0 pseudosection with one positive double point. Blow up at this double point to get an embedded 2-sphere C of self-intersection -5 in $E(1)_L\#\overline{\mathbf{CP}}^2$. The blowup formula [6] implies that $\mathcal{SW}_{E(1)_L\#\overline{\mathbf{CP}}^2}=(t^{-1}-t)(e+e^{-1})$ where e is is the class in the group ring corresponding to the new exceptional curve. Hence, the basic classes of $E(1)_L\#\overline{\mathbf{CP}}^2$ are $\pm F\pm E$. Now $\pm(F+E)\cdot C=\pm 3$ whereas $\pm(F-E)\cdot C=\mp 1$. It follows from the Rational Blowdown Theorem that only the basic classes $\pm(F+E)$ descend to the rational blowdown Q. Thus, Q has two basic classes whose Seiberg–Witten invariants are those of $\pm(F+E)$ in $E(1)_L\#\overline{\mathbf{CP}}^2$, namely, ∓ 1. $\square$

THEOREM 5.3. *There are infinitely many pairwise nondiffeomorphic* 4*-manifolds homeomorphic to* $P_8=\mathbf{CP}^2\#8\overline{\mathbf{CP}}^2$ *obtained from* $1/n$*-surgery on the nullhomologous torus* Λ *in* R*.*

Proof. For $n \geqslant 2$, let X_n be the 4-manifold obtained from $1/n$-surgery on Λ in R. By this we mean S^1 times $1/n$-surgery on λ in Figure 11. It is easy to see that X_n is simply connected and that there is an isomorphism $\varphi : H_2(X_n; \mathbf{Z}) \to H_2(R; \mathbf{Z})$, which is realized outside of a neighborhood of the surgery by the identity map. Thus, X_n is homeomorphic to P_8.

Morgan, Mrowka, and Szabó have calculated the effect of such a surgery on Seiberg–Witten invariants [14]. Given a class $k \in H_2(X_n)$,

$$\mathrm{SW}_{X_n}(k) = \mathrm{SW}_R(\varphi(k')) + n \sum_i \mathrm{SW}_Q(k'' + i[\Lambda_0]).$$

(Recall that the torus Λ is nullhomologous in R and the corresponding torus Λ_n, the core of the surgery, is nullhomologous in X_n.) Further, $k'' \in H_2(Q)$ is any class that agrees with the restriction of k in $H_2(R \smallsetminus \Lambda \times D^2, \partial)$ in the following diagram.

$$\begin{array}{ccc} H_2(Q) & \longrightarrow & H_2(Q, \Lambda_0 \times D^2) \\ & & \big\downarrow \cong \\ & & H_2(R \smallsetminus \Lambda \times D^2, \partial) \\ & & \big\uparrow \cong \\ H_2(X_n) & \longrightarrow & H_2(X_n, \Lambda_n \times D^2) \end{array}$$

The Seiberg–Witten invariants of the two $b^+ = 1$ manifolds X_n and R are calculated in corresponding chambers.

Given $k \in H_2(X_n)$ and H an element of positive self-intersection in $H_2(X_n)$, the small perturbation chamber, that is, the sign $\pm$ such that $\mathrm{SW}_{X_n}(k) = \mathrm{SW}^{\pm}_{X_n,H}(k)$ is determined homologically. This means that the small perturbation chambers for k in X_n and for $\varphi(k)$ in R correspond under φ. According to the previous theorem, there are only two classes: $\pm T$, $T = [S^1 \times m]$ in Q with nontrivial Seiberg–Witten invariants, and $\mathcal{SW}_Q(\pm T) = \mp 1$.

Thus, we have

$$\mathcal{SW}_{X_n} = \mathcal{SW}_R + \mathcal{SW}_Q = 0 + n\,(t^{-1} - t).$$

This shows that the manifolds X_n are pairwise nondiffeomorphic. That they are all minimal follows from the blowup formula. □

6. *Infinite families with* $b^- = 5, 6, 7$

For $b^- = 6, 7$, the constructions and calculations to show that there is a nullhomologous torus Λ in $R = R_6$ or R_7 such that $1/n$-surgery on Λ in R produces infinitely many distinct smooth structures on P_6 or P_7 are completely analogous to those in the last section. What differs is the proof that the Seiberg–Witten invariants of $R = R_6$ or R_7 vanish. We shall accomplish this by using an argument adapted from [15]. The important point in the argument is that we are starting our construction with $E(1)_K = E(1)$; so all the exceptional curves are represented by spheres, etc. First consider the $b^- = 7$ case. The classes

$$\begin{array}{ll} V_1 = H - E_1 - E_2 - E_3, & V_2 = H - E_2 - E_3 - E_4 \\ V_3 = H - E_3 - E_4 - E_5, & V_4 = H - E_6 - E_7 - E_8 - E_9 \\ V_5 = F - E_5, & V_6 = E_{11} - E_1 - E_2 \\ V_7 = E_{10} - E_1 - E_2, & V_8 = 2H - 3E_{11} \end{array}$$

are all orthogonal to the configuration C_5 and generate $H_2(P_{11} \smallsetminus C_5; \mathbf{Z}) = H_2(R_7 \smallsetminus B_5; \mathbf{Z})$. In P_{11}, the classes V_1, V_2, V_3 are represented by embedded spheres of self-intersection -2, V_4, V_6, V_7 are represented by embedded spheres of self-intersection -3, V_5 is represented by an embedded torus of self-intersection -3, and V_8 is represented by an embedded torus with square -5. According to the argument of [15], any basic class k of R_7 must satisfy the adjunction inequality:

$$V_i^2 + |k \cdot V_i| \leqslant 0, \quad i = 1, \ldots, 8.$$

Furthermore, k must satisfy

$$k^2 \geqslant 2, \quad k^2 \equiv 2 \pmod 8.$$

These follow from the fact that for any basic class, its corresponding moduli space must have nonnegative even dimension.

Since $H^2(R_7, \mathbf{Z})$ injects into $H^2(R_7 \smallsetminus B_5; \mathbf{Z})$, any Seiberg–Witten basic class of R_7 is uniquely determined by its intersection numbers with $V_1, \ldots, V_8$. There is now a finite check for possible basic classes k of R_7 which must satisfy these three previous conditions. This check turns up 40 classes in $H^2(R_7 \smallsetminus B_5; \mathbf{Z})$. Another class in $H_2(P_{11}; \mathbf{Z})$ which is orthogonal to C_5 is $V_9 = 2H - 3E_{10}$. It is represented by an embedded torus of self-intersection -5. Any basic class of R_7 must also satisfy the adjunction inequality with respect to V_9. This condition reduces the number of possible classes to 14. According to the Rational Blowdown Theorem, the Seiberg–Witten invariant of any such class is determined by the Seiberg–Witten invariant of an appropriate lift to P_{11}. Such a lift determines a Seiberg–Witten moduli space for P_{11} whose formal dimension is the same as that of the moduli space for R_7 corresponding to the class being lifted. This is accomplished via an extension across C_5 for each of the 14 possibilities.

The class $\hat{H} = V_1 + V_2 + V_3 + V_8$ is orthogonal to C_5 and $\hat{H}^2 = 4 > 0$, $\hat{H} \cdot H = 6 > 0$. Hence, $\hat{H}$ serves as a period point for R_7. Since $\mathcal{SW}_{P_{11},H} = 0$, for any characteristic cohomology class k of P_{11}:

$$\mathcal{SW}_{P_{11},\hat{H}}(k) = \begin{cases} 0, & \text{if the signs of } k \cdot \hat{H} \text{ and } k \cdot H \text{ agree} \\ \pm 1, & \text{if the signs of } k \cdot \hat{H} \text{ and } k \cdot H \text{ do not agree.} \end{cases}$$

Using this criterion on each of the 14 possibilities mentioned above shows that each has Seiberg–Witten invariant equal to 0. Thus, we have $\mathcal{SW}_{R_7} = 0$.

The $b^- = 6$ case follows similarly, but the calculation turns out to be easier. The classes

$$V_1 = E_{12} - E_1,\ V_2 = E_{11} - E_1,\ V_3 = E_{10} - E_1,\ V_4 = E_3 - E_1,$$
$$V_5 = E_2 - E_1,\ V_6 = H - 3E_1,\ V_7 = 2F + H - E_3 - E_{10} - E_{11} - E_{12}$$

generate $H_2(P_{12} \smallsetminus C_7; \mathbf{Z}) = H_2(R_6 \smallsetminus B_7; \mathbf{Z})$. In P_{12}, the classes $V_1, \ldots, V_5$ are all represented by embedded spheres of self-intersection -2, V_6 is represented by an embedded torus of self-intersection -8, and V_7 is represented by an embedded surface of genus 4 with square 5. If k is a basic class of X, then it must satisfy the adjunction inequality with respect to the classes $V_1, \ldots, V_7$; that is

$$V_i^2 + |k \cdot V_i| \leqslant 0, \quad i = 1, \ldots, 6; \qquad V_7^2 + |k \cdot V_i| \leqslant 2g - 2 = 6$$

and as above, k must also satisfy

$$k^2 \geqslant 3, \qquad k^2 \equiv 3 \pmod 8.$$

This time a check turns up no classes in $H^2(R_6 \smallsetminus B_7; \mathbf{Z})$ which satisfy these conditions. Hence, $\mathcal{SW}_{R_6} = 0$.

To obtain families of manifolds homeomorphic to P_5, we start with an elliptic fibration on $E(1)$ with two nodal fibers, two I_2-fibers and an I_6-fiber (gain, see [**17**]). This time we have two double node neighborhoods. One goes through the same construction in each of these to obtain an immersed pseudosection with two double points. Blowing up each, we obtain a sphere of square -9 in $E(1)\#2\overline{\mathbf{CP}}^2$, and adding on five of the spheres of the I_6-singularity, gives a copy of the configuration C_7, which can be rationally blown down to obtain a manifold R_5 homeomorphic to P_5. This process can be carried out so the the $+1$-torus of the proof of Proposition 5.1 descends to R_5; so we see that $\mathcal{SW}_{R_5} = 0$. Furthermore, we get the nullhomologous tori Λ_1, Λ_2, as before; one in each neighborhood. Performing 0-surgery on each gives a manifold Q with $b^+ = 3$ and $\mathcal{SW} = (t_1^{-1} - t_1)(t_2^{-1} - t_2)$. Perform $+1$-surgery on Λ_2 and $1/n$-surgery on Λ_1 to obtain Y_n which is homeomorphic to P_5, but which has $\mathcal{SW}_{Y_n} = n(t_1^{-1} - t_1)(t_2^{-1} - t_2)$.

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Ronald Fintushel
Department of Mathematics
Michigan State University
East Lansing, MI 48824
USA

ronfint@math.msu.edu

Ronald J. Stern
Department of Mathematics
University of California
Irvine, CA 92697
USA

rstern@uci.edu