RESOLUTIONS OF HOMOLOGY MANIFOLDS: A CLASSIFICATION THEOREM

ALLAN L. EDMONDS† AND RONALD J. STERN†

1. *Introduction*

Let BPL and BH denote the classifying spaces for stable PL block bundles [9] and stable homology cobordism bundles ([4], [5]). There is a natural map $j : BPL \to BH$ with homotopy fibre denoted by H/PL. If M is a closed (integral) homology manifold, a *resolution* of M is a pair (P, f), where P is a piecewise linear (PL) manifold and $f: P \to M$ is a surjective PL map which is acyclic, i.e. $\tilde{H}_*(f^{-1}(x)) = 0$ for all $x \in M$. Let $\tau : M \to BH$ classify the homology tangent bundle of M. We prove the following theorems.

EXISTENCE THEOREM. *There is a resolution of M if and only if τ lifts through j to BPL.*

CLASSIFICATION THEOREM. *The set of concordance classes (naturally defined) of resolutions of M is in one-to-one correspondence with the set of vertical homotopy classes of lifts of τ through j to BPL.*

The Existence and Classification Theorems are deduced using the

PRODUCT STRUCTURE THEOREM. *$M \times [-1, 1]^k$ is resolvable if and only if M is resolvable.*

Let θ_3^H denote the abelian group obtained from the set of oriented 3-dimensional PL homology spheres using the operation of connected sum, modulo those which bound acyclic PL 4-manifolds. Then according to N. Martin [3] (also see [7]),

$$\pi_i(H/PL) = \begin{cases} \theta_3^H & \text{if } i = 3 \\ 0 & \text{otherwise.} \end{cases}$$

Thus, using standard obstruction theory, the Existence Theorem can be restated in the form of the usual resolution theorem due to Sullivan [12] and Cohen [2] (also see [3] and [10]): M is resolvable if and only if a class in $H^4(M; \theta_3^H)$ is zero. Similarly, if M is resolvable, then there is a one-to-one correspondence between the set of concordance classes of resolutions of M and $H^3(M; \theta_3^H)$. This version of the Classification Theorem has also been obtained by N. Martin.

In §2 we summarize some basic facts about homology manifolds and their resolutions; in §3 we prove the Product Structure Theorem; in §4 we investigate the tangential properties of resolutions; and, finally, in §5 we complete the proofs of the main theorems.

We thank Clint McCrory and David Galewski for useful discussions concerning homology manifolds and resolutions.

Received 3 April, 1974; revised 11 December, 1974.

† Supported in part by National Science Foundation grant GP3641X1 while at the Institute for Advanced Study, Princeton, New Jersey.

2. *Background*

A compact polyhedron M is called a *homology n-manifold* if there is a triangulation K of M such that for any $x \in M$ and subdivision K_1 of K such that x is a vertex, $H_*(\mathrm{Link}\,(x, K_1))$ is isomorphic either to $H_*(S^{n-1})$ or to $H_*(\text{point})$. The *boundary* of M, ∂M, is the set of points x such that $H_*(\mathrm{Link}\,(x)) = H_*(\text{point})$ and is a closed (compact without boundary) homology $(n-1)$-manifold. We refer the reader to [6] for the basic properties of homology manifolds.

Two closed homology n-manifolds M and N are said to be *H-cobordant* if there is a homology $(n+1)$-manifold W such that ∂W is the disjoint union of M and N and $H_*(W, M) = H_*(W, N) = 0$. If M and N are homology n-manifolds with boundary, they are said to be H-cobordant if there is a homology $(n+1)$-manifold W such that $\partial W = M \cup W_0 \cup N$ where W_0 is an H-cobordism from ∂M to ∂N and $H_*(W, M) = H_*(W, N) = 0$.

Let M be a homology manifold and N be a codimension zero PL submanifold of ∂M. A *resolution of M rel N* is a pair (P, f), where P is a PL manifold and $f\colon (P, \partial P) \to (M, \partial M)$ is a surjective PL map of pairs such that (i) $f^{-1}(\partial M) = \partial P$, (ii) $f \mid f^{-1}(N)$ is a PL homeomorphism, and (iii) f is acyclic, i.e. $H_*(f^{-1}(x)) = 0$ for all $x \in M$. If (P, f) is a resolution of M and X is a subcomplex of M, then $f \mid f^{-1}(X)$: $f^{-1}(X) \to X$ is an acyclic map and so, by the Vietoris–Begle Mapping Theorem [**11**; p. 344], induces an isomorphism $H_*(f^{-1}(X)) \to H_*(X)$.

Two such resolutions (P_i, f_i), $i = 0, 1$, of M rel N are *concordant rel N* if there is a resolution (Q, F) of $M \times I$ rel $N \times I$ such that $(F^{-1}(M \times i), F \mid F^{-1}(M \times i))$ is (P_i, f_i) for $i = 0, 1$. If $\partial M = N = \varnothing$, we denote the set of concordance classes of resolutions of M by Res (M).

Sullivan [**12**] and Cohen [**2**] (see also Martin [**3**] and Sato [**10**]) have constructed an elegant obstruction theory for resolving a homology manifold. The theory shows that if N is a codimension zero PL submanifold of the boundary of a homology manifold M, then there is an element σ_M in $H^4(M, N; \theta_3^H)$ (θ_3^H as defined in §1) such that $\sigma_M = 0$ if and only if M can be resolved rel N. See especially Martin [**3**] for this relative formulation.

We shall also use the elementary fact that the mapping cylinder of a surjective PL acyclic map between two homology manifolds is an H-cobordism and in particular a homology manifold.

For the basic properties of homology cobordism (disk) bundles over homology manifolds, we refer the reader to [**4**]. Recall from [**4**] that if ξ is a homology cobordism bundle over a homology manifold M, then the total space $E(\xi)$ is also a homology manifold. Also if ξ and ζ are equivalent homology cobordism bundles then $E(\xi)$ and $E(\zeta)$ are H-cobordant as homology manifolds.

Martin and Maunder in [**4**] and [**5**] have shown that there is a space BH which classifies stable equivalence classes of homology cobordism bundles. There is a natural map $j\colon BPL \to BH$, where BPL denotes the classifying space for stable PL block bundles [**9**]. We make j into a fibration and call its fibre H/PL. It is not hard to see that two homology cobordism bundles ξ and ζ are stably equivalent if and only if $\xi \times I^m$ is isomorphic to $\zeta \times I^n$, for some m and n, in the sense of [**4**].

If M is a homology manifold we denote its homology tangent bundle by $\tau(M)$. If M is a PL manifold, we denote its PL tangent block bundle by $T(M)$. In either case, the tangent bundle is given by a regular neighbourhood of the diagonal in $M \times M$.

If ξ is a homology cobordism bundle (resp. PL block bundle) over a complex X, we will also often denote by $\xi : X \to BH$ (resp., $\xi : X \to BPL$) the stable classifying map of the bundle.

Finally we recall the construction of the pullback of a homology cobordism bundle [5; p. 112]. Let $f: M \to N$ be a simplicial map of homology manifolds and let ξ be a homology cobordism bundle over N. Let C_f denote the simplicial mapping cylinder of f. Then, as in [4; 3.5], ξ can be extended to a homology cobordism bundle ζ over all of C_f and ζ is unique up to isomorphism. Then we define $f^*\xi$ to be $\zeta \mid M$.

3. *The Product Structure Theorem.*

Let M be a homology manifold such that ∂M is a PL manifold. Our goal in this section is to relate the resolutions of $M \times B^k$ ($B^k = [-1, 1]^k$) with those of M.

THEOREM 3.1 (Product Structure Theorem). *Let (Q, g) be a resolution of $M \times B^k$* rel $\partial M \times B^k$. *Then there is a resolution (P, f) of M, where P is a* PL *submanifold of Q with trivial normal block bundle, and a commutative diagram*

$$\begin{tikzcd} Q \arrow[r, "g"] & M \times B^k \\ P \arrow[u] \arrow[r, "f"] & M \arrow[u, "h"'] \end{tikzcd}$$

where h is a proper PL *embedding isotopic to the standard embedding $M \times 0$.*

Proof. A finite induction shows that it suffices to find a resolution (P, f) of $M \times B^{k-1}$ rel $\partial M \times B^{k-1}$, where P is a properly embedded PL submanifold of Q with trivial normal bundle, and a commutative diagram

$$\begin{tikzcd} Q \arrow[r, "g"] & M \times B^k \\ P \arrow[u] \arrow[r, "f"] & M \times B^{k-1} \arrow[u, "h"'] \end{tikzcd}$$

in which h is a proper embedding properly isotopic to the standard inclusion

$$M \times B^{k-1} \subsetneq M \times B^k.$$

To this end we may assume that Q and $M \times B^k$ are triangulated so that g is simplicial and so that the standard $M \times B^{k-1}$ is a full subcomplex of $M \times B^k$. Let N be the simplicial neighbourhood of $M \times B^{k-1}$ in the first derived subdivision of $M \times B^k$. Since inverse images of dual cells by simplicial maps are manifolds [1; 5.6], we see that $Q_0 = g^{-1}(N)$ is a codimension zero submanifold of Q and that $g \mid Q_0 : Q_0 \to N$ is a resolution rel $N \cap (\partial M \times B^k)$. Now the frontier of N breaks into two pieces; let N^+ be one of them. Let $P = g^{-1}(N^+)$, which must be one of the two components of the frontier of Q_0 in Q. Let $f = g \mid P$. Then the theory of derived neighbourhoods shows that there is a proper PL homeomorphism

$$(M \times B^{k-1}, \partial M \times B^{k-1}) \to (N^+, N^+ \cap \partial M \times B^k)$$

which is isotopic to the standard inclusion $M \times B^{k-1} \subset M \times B^k$. Finally, P has trivial normal bundle in Q, being a boundary component of a codimension zero submanifold.

4. *Tangential Properties of Resolutions*

If M is a homology manifold and ξ is a homology cobordism bundle over M, then a (stable) *reduction* of ξ to a PL block bundle is a homology cobordism bundle η over $M \times I$ such that $\eta \mid M \times 0 = \xi$ stably and $\eta \mid M \times 1$ is a PL block bundle. Two reductions of ξ are *equivalent* if there is a reduction of $\xi \times I$ over $(M \times I) \times I$ between them. Let $H/PL(\xi)$ denote the set of equivalence classes of stable reductions of ξ.

Also, let $\mathrm{Lift}(\xi)$ denote the set of vertical homotopy classes of lifts to BPL of $\xi : M \to BH$ through $j : BPL \to BH$. Then the following lemma is an exercise in the definitions, using the homotopy lifting property.

LEMMA 4.1. *There is a one-to-one correspondence between the elements of* $H/PL(\xi)$ *and those of* $\mathrm{Lift}(\xi)$.

LEMMA 4.2. *If N and M are homology manifolds and $f : N \to M$ induces an isomorphism on homology, then the natural map*

$$f^* : H/PL(\tau(M)) \to H/PL(f^* \tau(M))$$

is a bijection.

Proof (of 4.2). If η is a reduction of $\tau(M)$, $f^*[\eta]$ is given by $[(f \times 1)^*\eta]$. Obstruction theory shows that there are lifts of $\tau(M)$ to BPL if and only if there are lifts of $f^*\tau(M)$ to BPL. So assume such lifts exist and choose one lift α of $\tau(M)$. Then comparison with α and with $f^*\alpha$ and application of the homotopy lifting property for fibrations shows that we have a commutative diagram

$$\begin{array}{ccc}
H/PL(\tau(M)) & \longrightarrow & H/PL(f^* \tau(M)) \\
\downarrow & & \downarrow \\
\mathrm{Lift}(\tau(M)) & \longrightarrow & \mathrm{Lift}(f^* \tau(M)) \\
\downarrow & & \downarrow \\
[M; H/PL] & \longrightarrow & [N; H/PL]
\end{array} ,$$

where horizontal arrows are induced by f and the vertical arrows are bijections. But the lower horizontal arrow is a bijection by obstruction theory (all coefficients are simple since $\pi_1(H/PL) = 0$), completing the proof.

THEOREM 4.3. *A resolution (P, f) of a homology manifold M determines a well defined element of $H/PL(\tau(M))$ which depends only on the class of (P, f) in* Res (M).

Proof. The mapping cylinder C_f of f is a homology manifold (see §2). Let $\pi : P \times I \to C_f$ be the natural quotient map. Then $\pi^* \tau(C_f)$ is a stable reduction of $f^* \tau(M)$ to the PL block bundle $T(P)$. Using (4.2), let α be the unique element of $H/PL(\tau(M))$ corresponding to $f^*\tau(M)$. The same construction applied to a concordance shows that the class of α in $H/PL(\tau(M))$ depends only on the concordance class of (P, f) in Res (M).

By (4.3) we obtain a well-defined function

$$\psi : \mathrm{Res}\,(M) \to H/PL(\tau(M)).$$

5. *Bijectivity of* ψ.

We first show that the existence of a reduction of $\tau(M)$ to a PL block bundle implies that M is resolvable. Carefully done this shows that ψ is surjective. A suitable relative version of the above then shows that ψ is injective.

If (P, f) is a resolution of M let $[P, f]$ denote its class in Res (M); if η is a stable reduction of $\tau(M)$ to a PL block bundle, let $[\eta]$ denote its class in $H/PL(\tau(M))$.

THEOREM 5.1 (Existence Theorem). *Let M be a compact homology n-manifold with $\partial M = \varnothing$. If η is a stable reduction of $\tau(M)$ to a* PL *block bundle, then there is a resolution (P, f) of M such that $\psi[P, f] = [\eta]$.*

Proof. Let M be embedded in some Euclidean space $\mathbb{R}^k$ and let N be a closed regular neighbourhood of M in $\mathbb{R}^k$. Then N is a parallelizable PL k-manifold and there is a PL deformation retraction $r : N \to M$. The induced stable reduction $\xi = (r \times 1)^*\eta$ of $r^*\tau(M)$ to a PL block bundle is a homology cobordism bundle over $N \times I$ such that $\xi_0 = \xi \mid N \times 0$ is a PL block bundle and $\xi_1 = \xi \mid N \times 1$ is stably isomorphic to $r^*\tau(M)$. The total space $E(\xi)$ of ξ is an H-cobordism between the PL manifold $E(\xi_0)$ and the homology manifold $E(\xi_1)$. Also $E(\xi_1)$ is PL homeomorphic to $M \times I^m$ for some m, since $E(\xi_1) = E(r^*\tau(M)) \times I^q$ for some q, and $E(r^*\tau(M))$ can be seen to be a regular neighbourhood of $M \times 0$ in $M \times \mathbb{R}^k$ (compare [**8**; 5.12]).

Since $H^*(E(\xi), E(\xi_0)) = 0$, the obstruction to resolving $E(\xi)$ rel $E(\xi_0)$ is zero, so let (Q, g) be such a resolution. Pulling back a small regular neighbourhood of $E(\xi_1) \cong M \times I^m$ in $\partial E(\xi)$ we obtain by restriction, as in the proof of the Product Structure Theorem, a resolution (R, h) of $M \times I^m$, and hence a resolution (P, f) of M.

It remains to see that $\psi[P, f] = [\eta]$. For this we use the fact that we resolved the entire H-cobordism $E(\xi)$. Now by (4.3), (Q, g) determines a reduction of $\tau(E(\xi))$ to a PL block bundle, hence a reduction of $\tau(E(\xi)) \mid N \times I$ to a PL block bundle. But $\tau(E(\xi)) \mid N \times I \approx \tau(N \times I) \oplus \xi$ which stably is just ξ since N is parallelizable. Thus (Q, g) determines a stable reduction β over $(N \times I) \times I$ with $\beta \mid N \times I \times 1$ a PL block bundle and $\beta \mid N \times I \times 0 = \xi$. Furthermore, $\beta \mid N \times 1 \times I$ is a PL block bundle, being obtained from the trivial reduction induced by the identity map and the isomorphism $\tau(E(\xi)) \mid N \times 1 = T(E(\xi) \mid N \times 1)$. Also, $\beta \mid N \times 0 \times I$ is stably $r^*\mu$ where μ is the reduction of $\tau(M)$ induced by (P, f). This follows because, by the Product Structure Theorem, P has a trivial normal PL block bundle in Q and there is a commutative diagram

$$\begin{array}{ccc} Q & \xrightarrow{g} & M \times I^m \\ \uparrow & & \uparrow h \\ P & \xrightarrow{f} & M \end{array}$$

where h is a PL embedding which is PL isotopic to the standard embedding $M \times 0$.

Thus β determines an equivalence between the stable reductions $\xi = r^*\eta$ and $r^*\mu$ and hence between η and μ by (4.2) as desired.

In a similar fashion one proves the following relative version of (5.1).

THEOREM 5.2. *Let M be a homology manifold with ∂M a* PL *manifold. If η is a stable reduction of $\tau(M)$ to a* PL *block bundle such that $\eta \mid \partial M \times I = T(\partial M) \times I$ as stable* PL *block bundles, then there is a resolution (P, f) of M* rel ∂M.

COROLLARY 5.3. (Classification Theorem). *If M is a homology manifold with $\partial M = \emptyset$, then ψ :* Res $(M) \to H/PL(\tau(M))$ *is bijective. Hence* Res $(M) \approx$ Lift $(\tau(M))$.

Proof. Surjectivity follows from (5.1). To prove injectivity, let (P_i, f_i), $i = 0, 1$, be two resolutions of M such that the corresponding stable reductions η_0 and η_1 are equivalent. Then we obtain a homology cobordism bundle η over $M \times I \times I$ such that $\eta \mid M \times I \times i = \eta_i$ stably, for $i = 0, 1$, $\eta \mid M \times 1 \times I$ is a PL block bundle, and $\eta \mid M \times 0 \times I$ is $\tau(M) \times I$ stably.

Let C_i denote the mapping cylinder of f_i, $i = 0, 1$, and consider the homology manifold $N = C_0 \cup M \times I \cup C_1$, where we identify the zero ends of C_0 and C_1 with $M \times 0$ and $M \times 1$ respectively, Using the natural retraction of N onto $M \times I$, we obtain from η a reduction of $\tau(N)$ to a PL bundle rel $P_0 \cup P_1$. Thus we obtain by (5.2) a resolution (Q, g) of N rel $\partial N = P_0 \cup P_1$.

Define a map $r : N \to M \times I$ by

$$r[x, t] = \begin{cases} (f_0(x), (1-t)/3) & \text{if} \quad [x, t] \in C_0, \\ (f_1(x), (2+t)/3) & \text{if} \quad [x, t] \in C_1, \\ (x, (1+t)/3) & \text{if} \quad [x, t] \in M \times I. \end{cases}$$

Then (Q, rg) is a resolution of $M \times I$ and a concordance between (P_0, f_0) and (P_1, f_1).

Using similar techniques one also proves the following more general relative classification theorem.

THEOREM 5.4. *Let M be a homology manifold with ∂M a* PL *manifold. Then there s a one-to-one correspondence between concordance classes of resolutions of M* rel ∂M *and vertical homotopy classes of lifts of $\tau(M) : M \to BH$ to BPL* rel *a fixed lift*

$$T(\partial M) : \partial M \to BPL \quad \text{of} \quad \tau(\partial M) : \partial M \to BH.$$

References

1. M. Cohen, " Simplicial structures and transverse cellularity ", *Ann. of Math.*, 85 (1967), 218–245.
2. ———, " Homeomorphisms between homotopy manifolds and their resolutions ", *Inv. Math.*, 10 (1970), 239–250.
3. N. Martin, " On the difference between homology and piecewise-linear bundles ", *J. London Math. Soc.*, 6 (1973), 197–204.
4. ——— and C. Maunder, " Homology cobordism bundles ", *Topology*, 10 (1971), 93–110.
5. C. Maunder, " On the Pontrjagin classes of homology manifolds ", *Topology*, 10 (1971), 111–118.
6. ———, *Algebraic topology* (Van Nostrand, London, 1970).
7. T. Matumoto and Y. Matsumoto, " The unstable difference between homology cobordism and piecewise linear block bundles ", Preprint.
8. J. Milnor, " Microbundles I ", *Topology* 3 suppl. 1 (1964), 53–80.
9. C. P. Rourke and B. J. Sanderson, " Block bundles I ", *Ann. of Math.*, 87 (1968), 1–28.
10. H. Sato, " Constructing manifolds by homotopy equivalences I. An obstruction to constructing PL manifolds from homology manifolds ", *Ann. Inst. Fourier, Grenoble*, 22 (1972), 271–286.

11. E. H. Spanier, *Algebraic topology* (McGraw-Hill, New York, 1966).
12. D. Sullivan, " Singularities in spaces ", in *Proceedings of Liverpool Singularities Symposium II*, Springer Lecture Notes 209, 1971.

Cornell University,
Ithaca, New York 14853,
U.S.A.

University of Utah,
Salt Lake City, Utah 84112
U.S.A.