Exotic Group Actions on Smooth 4-Manifolds

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Joint work with **Ron Fintushel** and **Nathan**

Exotica in dimension

Important consequences of Seiberg-Witten (and Donaldson) theory

Exotic smooth structures

- Existence of infinitely many distinct smooth structures on many topological

Exotic embeddings of

- Existence of infinitely many surfaces in a fixed smooth 4-manifold which are topologically but not smoothly equivalent

In higher dimensions corresponding results are up to FINITE

Conjecture: Every (simply-connected) topological 4-manifold has either no or infinitely many distinct smooth structures

General Question: Do smooth 4-manifolds have any finiteness properties?

Exotic smooth group actions?

Fix a smooth 4-manifold X and group G that acts smoothly on X. Are are there only finitely many actions of G on X that are equivariantly homeomorphic but not equivariantly

Some known example of

Example: Exotic involutions on S^4, Quotient = Fake RP^4
(F- Stern/ Cappell - Shaneson, Gompf)

Unknown if only finitely many since action is free and distinguished by smooth structure on orbit space

ISSUE: Want orientation-preserving

Ue's examples

1998. For any nontrivial finite group G there exists a smooth 4-manifold that has infinitely many free G-actions so that their orbit spaces are homeomorphic but mutually nondiffeomorphic.

The examples

Y: Q-homology S^4 with $\pi_1(Y) \to G$, onto, $\ni$ corr. cover is $\tilde{Y} = S^2 \times S^2 \# Z$, some Z. Get Y by spinning known 3D example.

$X_0 = E(2)_p$, $X_1 = E(2)_q$, $p \neq q$ odd (log transformed K3's)

The G-covers Q_i come from $\pi_1(X_i \# Y) \to \pi_1(Y) \to G$

$$Q_i \cong \tilde{Y} \# |G| X_i \cong S^2 \times S^2 \# Z \backslash \# |G| X_i$$

$S^2 \times S^2 \# Z \# |G| X_i \cong Q_j$ since the $E(2)_p$'s stabilize after one $\# S^2 \times S^2$.

Same issues as before: Action is free, so exoticness determined by orbit space; examples also reducible

Seek actions on irreducible 4-manifolds with fixed points

Warm-Up

Double cover of S^4 br. over standardly embedded torus = $S^2 \times S^2$

X: s.c. smooth 4-mfd, K: knot in S^3,

X_K: result of knot surgery

Double cover of X_K branched over std T^2 emb in ball

$\cong X_K \# S^2 \times S^2 \# X_K$

Auckly: $X_K \# S^2 \times S^2 \cong X \# S^2 \times S^2$

Varying K gives distinct involutions on same s.c 4-mfd

Issues: 'Distinct' because different orbit spaces.

Examples are reducible

Seek actions with fixed points on irreducible 4-manifolds and same orbit space

Strategy to produce cyclic actions on same manifold

- Fix an orbit space Y
- Take a fixed surface B in Y representing p times a homology class so associated p-fold branched cover X is a irreducible (say complex)
- modify branch surface so that it is topologically but not smoothly isotopic to B and so that cover remains diffeomorphic to X.
- Actions distinguished by branch curves not being isotopic.

Well - turns out you CAN do this
(with modest assumption on given branch curve B)

Exotic cyclic group actions

Theorem (Fintushel, Stern, Let Y be a simply connected 4-manifold with $b^+ \geq 1$ containing an embedded surface Σ of genus $g \geq 1$ with $\Sigma^2 \geq 0$.

Suppose that $\pi_1(Y-\Sigma)=Z_d$ and that the pair (Y, Σ) has a nontrivial relative Seiberg-Witten inv't. Suppose also that Σ contains a nonseparating loop which bounds an embedded 2-disk in $Y-\Sigma$. Let d' divide d, and let X be the (simply connected) d'-fold cover of Y branched over Σ.

Then X admits an infinite family of smoothly distinct but topologically equivalent actions of $Z_{d'}$.

Simple examples of complex surfaces with infinitely many actions of a fixed finite group

Curves in CP^2

$Y=CP^2$, Σ embedded degree d curve.

X = degree d hypersurface in CP^3

If d=3, $X=CP^2\#6\overline{CP}^2$ $\Rightarrow$ we have infinitely many smoothly inequivalent topologically equivalent Z_3-actions on $CP^2\#6\overline{CP}^2$.

If d=4, X=K3 $\Rightarrow$ smoothly inequivalent topologically equivalent Z_4-actions on the K3-surface.

Also theorem $\Rightarrow$ families of Z_2 and Z_3-actions on K3.

Z_k–actions, $k|d$, on degree d hypersurfaces in CP^3

Strategy to produce cyclic actions on same manifold

- Fix an orbit space X
- Take a fixed surface B in X representing p times a homology class so associated p-fold branched cover is a irreducible (say complex)
- modify branch surface B so that

✳ topologically but not smoothly isotopic to B

✳ cover remains diffeomorphic

How to modify branch surface B

How to modify branch surface B

Knot surgery

K: knot in S^3, T: square 0 essential torus in X

$X_K = X\text{-}N_T \cup S^1 \times (S^3 - N_K)$

$S^1 \times (S^3 - N_K)$ has the homology of $T^2 \times D^2$

Facts

- If X and X-T both simply connected, so is X_K.

  (So X_K homeo to X)

- $SW_{X_K} = SW_X \cdot \Delta_K(t^2)$

So changes smooth structure – not quite what we

Rim Surgery

$\Sigma \subset X$: emb orientable surface in $\pi_1=0$ 4-mfd.

C homologically essential loop in Σ

Rim torus: preimage of C in bdry of normal bundle of Σ

Rim surgery = knot surgery on rim torus.

Does not change X, but
can change embedding type of Σ. Get $\Sigma_K \subset X$.

Main Results on Effects of Rim Surgery

Th'm (Fintushel-S) Let $g(\Sigma)>0$. If $\pi_1(X)= 0=\pi_1(X-\Sigma)$, then there is a self-homeo of X throwing Σ_K on Σ. If $\Sigma^2>0$, then the relative SW-invariant of (X, Σ_K) is the relative SW-invariant of (X,Σ) times the Alexander polynomial of K.

Get smoothly inequivalent embeddings if original SW inv't is $\neq 0$. (E.g. symplectic submanifold.)

Relative SW-invariant lives in monopole Floer homology group

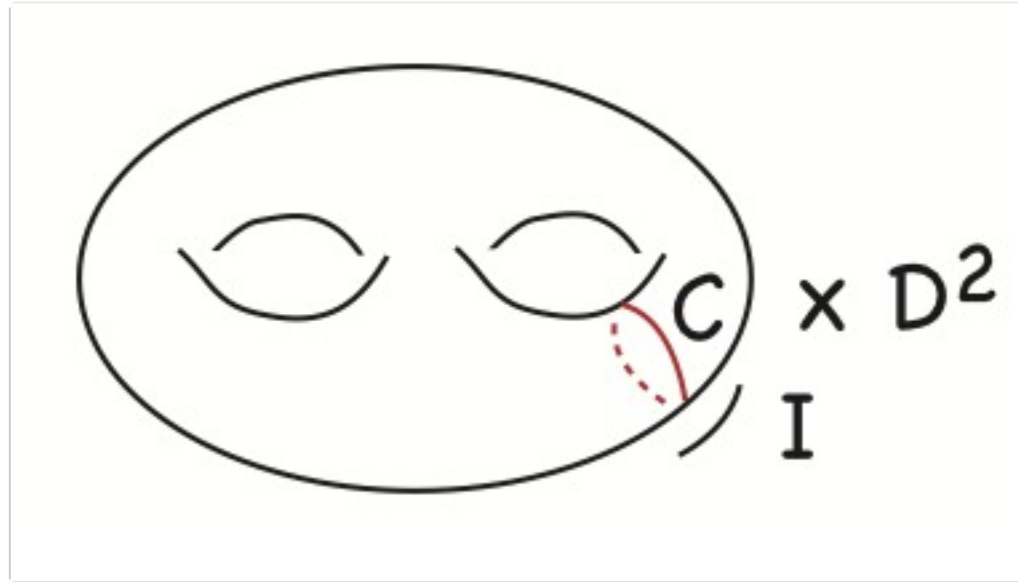
THERE IS A PROBLEM

Want to take cyclic branched covers.

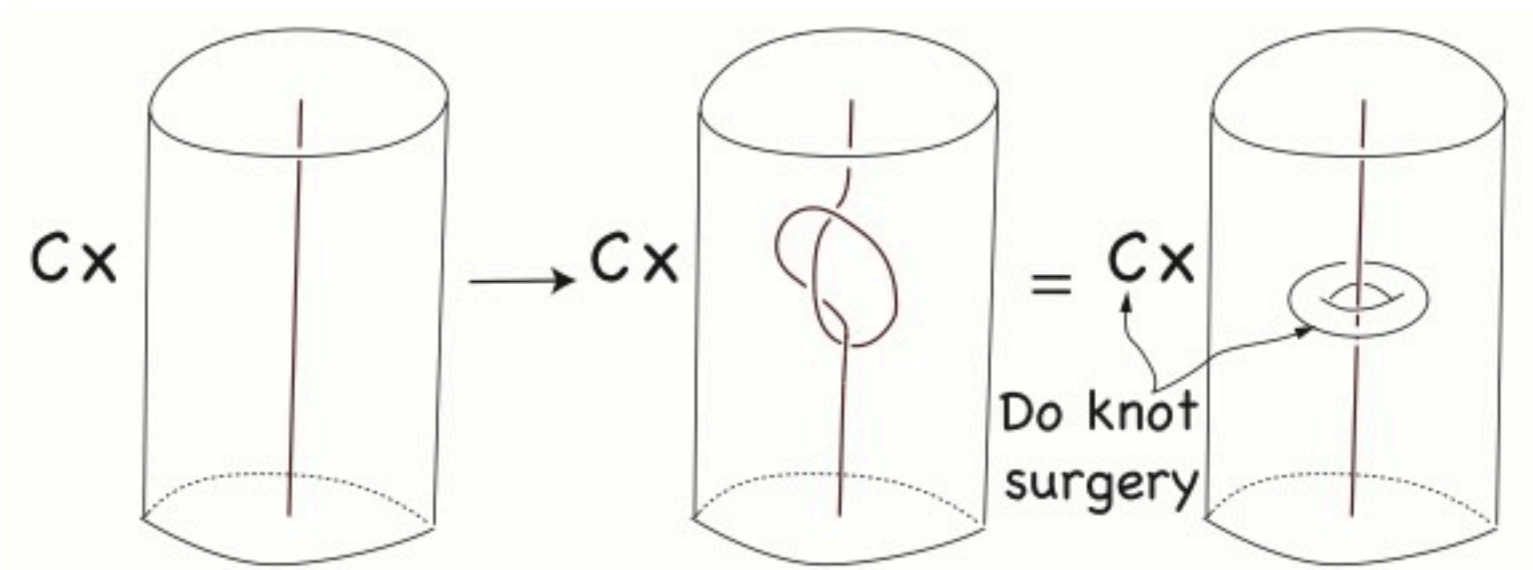
Need $\pi_1(X-\Sigma) = \mathbf{Z}_d$.

Problem: Rim surgery will not preserve this condition.

More on Rim Surgery



Effect of rim surgery:

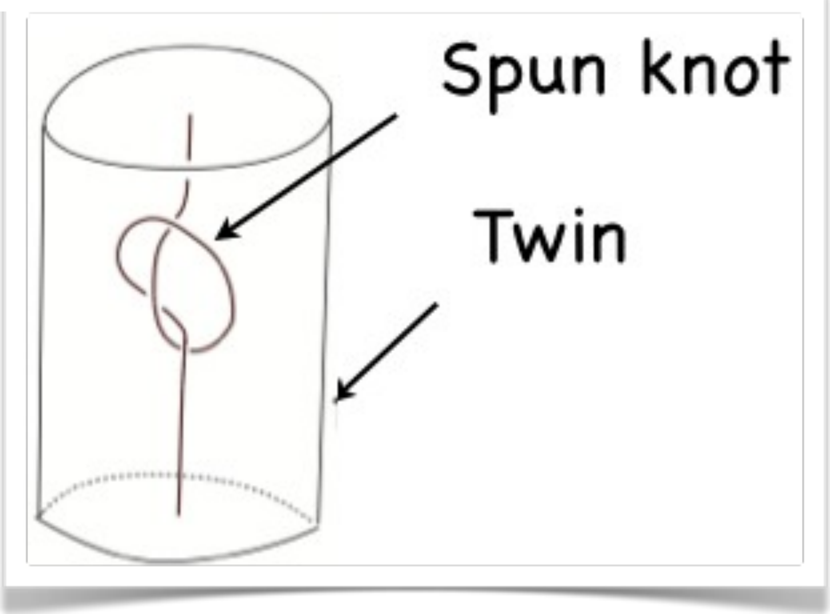


Another way to view rim surgery

Observation: The complement in S^4 of two 2–spheres intersecting in 2 points

Spinning a knot K in S^3 gives 2-knot in S^4:

S^1-action on S^4. Orbit space B^3.



Spun knot = preimage of knotted arc.

Preimage of ∂B^3= twin

Knot surgery replaces $C \times S^1 \times D^2$ =

$C \times$ Nbd(unknotted arc) with

S^4- ({spun knot}$\cup${twin})

An effective modification: Twist-Rim

Want to take cyclic branched covers.

Need $\pi_1(X-\Sigma) = \mathbf{Z}_d$.

Problem: Rim surgery will not preserve this condition.

Solution (Kim-Ruberman) k-Twist-spun rim surgery does preserve $\pi_1=\mathbf{Z}_d$ as long as k is prime to d.

New surface obtained is topologically equivalent to the old one in this case.

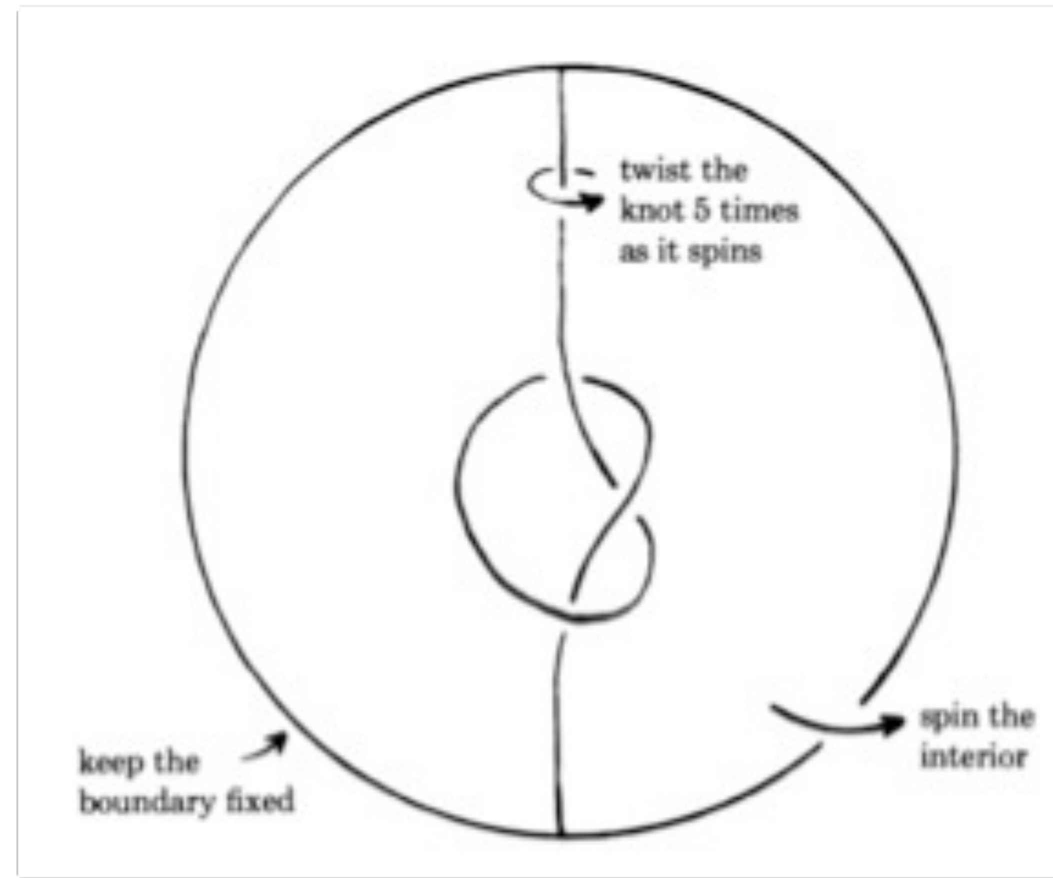
Relative SW-invariant is the same as for ordinary rim surgery.

Twist-spinning

Twist-spinning a knot

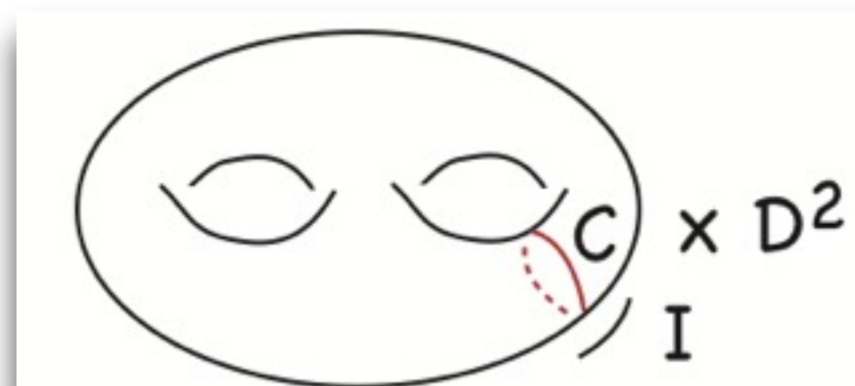
K: knot in S^3. Twist-spinning operation due to Zeeman.

Get knotted S^2 in S^4 and circle action.



Twist-spun rim surgery, $\Sigma_{K,k}$

Replace $C \times S^1 \times D^2$ with S^4-(twist-spun knot$\cup$twin)



$C \times I \times D^2$ replaced by complement of trivial twin in S^4.

Annulus on surface replaced by twist-spun knot minus polar caps.

Cyclic Group Actions

Y: simply connected smooth 4-manifold.

Σ genus ≥ 1 surface embedded in Y such that $\pi_1(Y-\Sigma)=Z_d$.

C: nonseparating loop on Σ, bounds disk in complement.

X = d-fold branched cyclic cover.

Choose k relatively prime to d. $\exists$ family of knots $K_i \ni$ d-fold branched covers X_i of $(Y, \Sigma_{K\_i,k})$ are all topologically equivalent but smoothly distinct.

$\Rightarrow$ Topologically equivalent but smoothly distinct actions of Z_d

Need to see that X_i are diffeomorphic to each other.

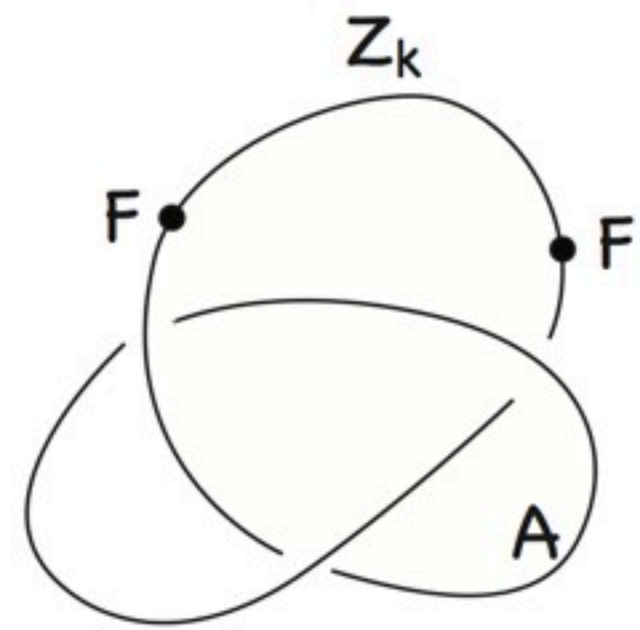
Circle Actions on S^4 and Twist-Spinning

S^1-action on S^4 determined up to equivariant diffeomorphism by orbit data.

Orbit space: B^3 or S^3
Fixed point set = S^2 or S^0
Exceptional orbit image = 0, 1, or (two arcs = circle)

Twist-spinning a knot

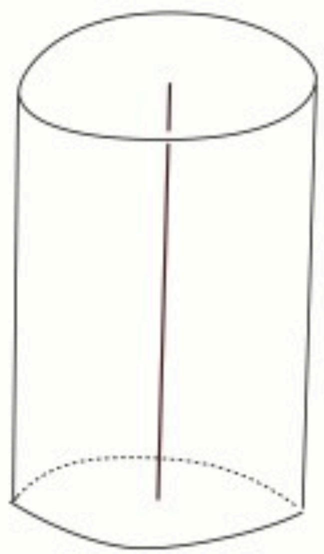
K: knot in S^3. S^1-action on S^4 with orbit space S^3, $p: S^4 \to S^3$ where the isotropy type corresponding to the arc A is trivial.



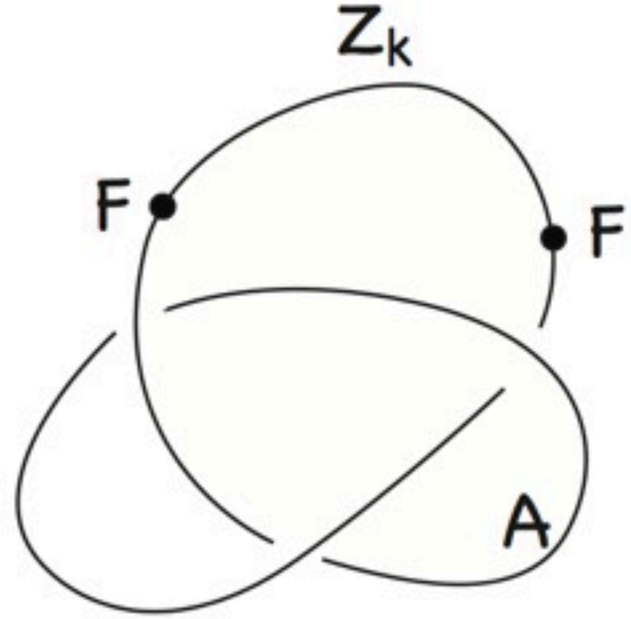
k-twist spin of K = $p^{-1}(\bar{A}) \subset S^4$

Twist-Spun Rim Surgery

To construct $\Sigma_{K,k}$

$C \times$ 

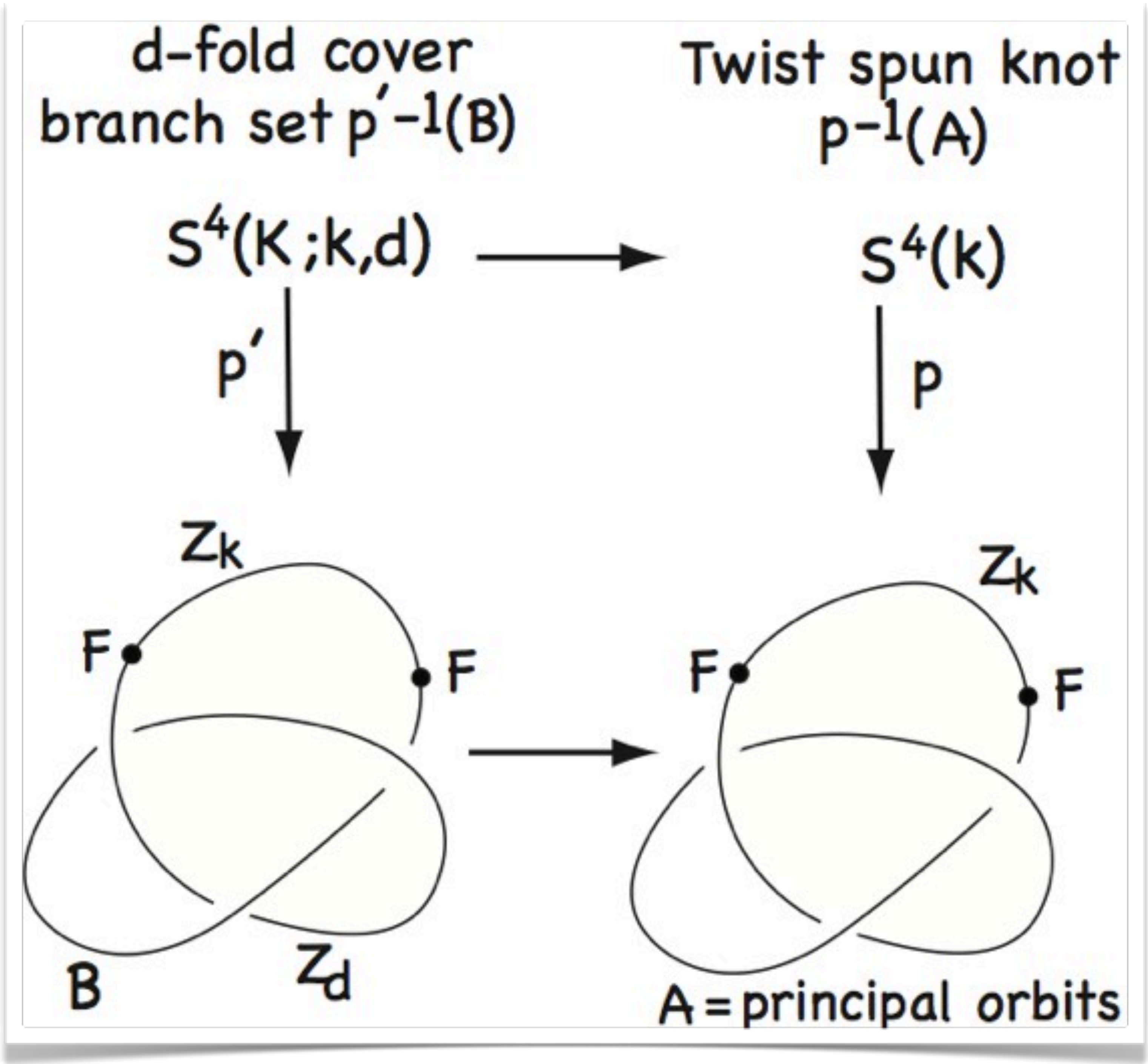
gets replaced with S^4-Nbd$(p^{-1}(E_k))$
where E_k = closed arc labeled 'Z_k'.



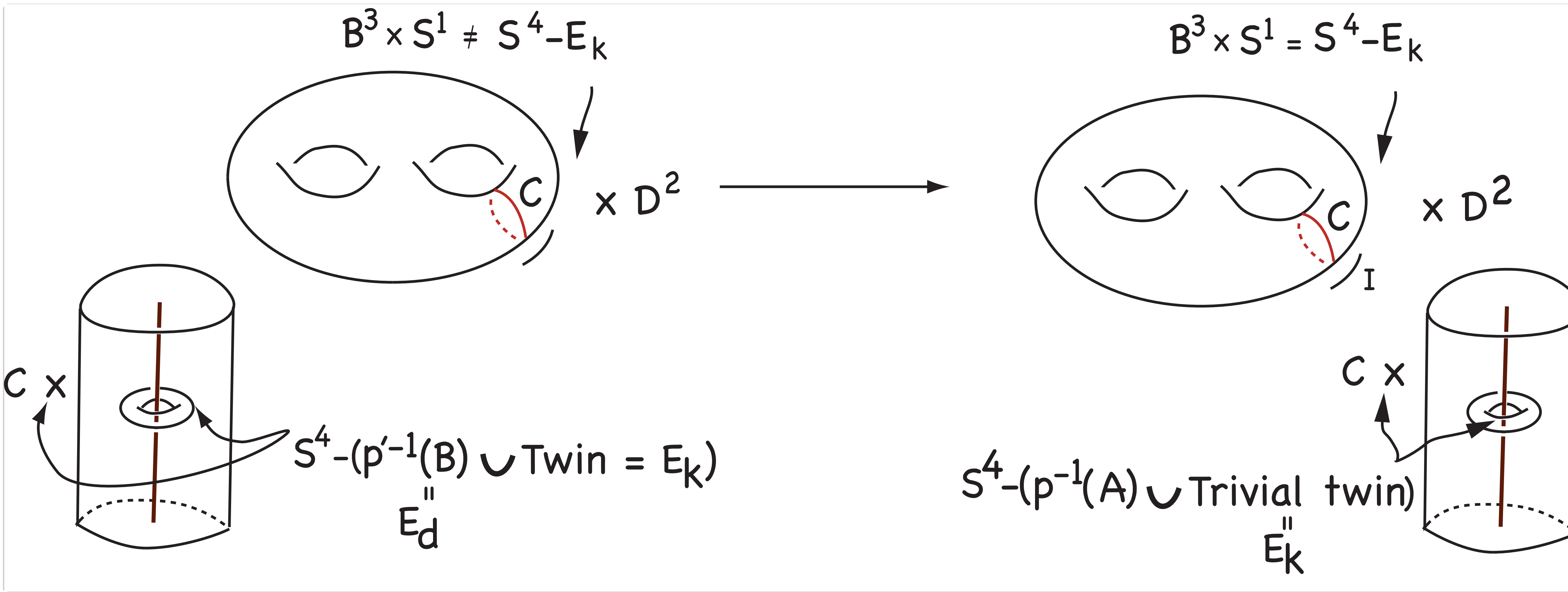
$$\subset \begin{array}{c} S^4 \\ \downarrow p \\ S^3 \end{array}$$

Branched Covers of Twist-Spun Knots

d-fold br cover of k-twist-spun knot



Covering Spaces



In cover, replacing $C \times I \times D^2$ with $S^4 - E_k \neq S^1 \times B^3$

C bounds disk, $C \times I \times D^2 \cup \text{Nbd(disk)} = B^4$ in X

After knot surgery in Y, B^4 in cover X

becomes $S^4(K;k,d) - B^4 = S^4 - B^4 = B^4$

$\Rightarrow X_K$ diffeomorphic to X

Next steps to seek finiteness properties for smooth 4-manifolds

Given a fixed smooth 4-manifold X and a self diffeomorphism f are there only finitely many diffeomorphisms topologically but not

The infinitely many distinct group actions in this talk provide infinitely many diffeomorphisms that are topologically isotopic - are they

Work in progress -what to expect???

Finiteness may be found when exploring symplectic properties of smooth 4-manifolds - but this is a different story

THE END

Thanks to the organizers

Have a safe journey home

Relative Seiberg-Witten Invariants

By blowing up, assume $\Sigma^2=0$

Seiberg-Witten invariant of $Y- N(\Sigma)$ obtained from spin^c-structures $\mathfrak{s}$ on Y satisfying $\langle c_1(\mathfrak{s}),\Sigma\rangle = 2g-2$

$$SW_{Y|\Sigma} : H_2(Y-N(\Sigma),\Sigma\times S^1;\mathbf{R})\to\mathbf{R}$$

(Kronheimer/Mrowka)

Role of basic classes played by $z\in\pi_0(B(Y-N(\Sigma);[a_0]))$

principal homogeneous space for $H^2(Y-N(\Sigma),\partial)$

z= solution of SW eq'ns.

a_0: unique spin^c-structure on $\Sigma\times S^1$ of degree $2g-2$