

The robustness of hybrid equilibria in costly signaling games

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Abstract Recent work on costly signaling games has identified new Nash equilibria in addition to the standard costly signaling equilibrium as a possible explanation for signaling behavior. These so-called hybrid equilibria are Liapunov stable, but not asymptotically stable for the replicator dynamics. Since some eigenvalues of the hybrid equilibria have zero real part, this result is not structurally stable. The purpose of this paper is to show that under one reasonable perturbation of the replicator dynamics—the selection-mutation dynamics—rest points close to the hybrid equilibrium exist and are asymptotically stable. Moreover, for another plausible version of the replicator dynamics—Maynard Smith’s adjusted replicator dynamics—the same is true. This reinforces the significance of hybrid equilibria for signaling.

Keywords Costly signaling games · hybrid equilibrium · replicator dynamics · structural stability · selection-mutation dynamics

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1 Introduction

The question of how honest signaling is possible in the face of conflicts of interest has been investigated by many biologists and economists. In biology, Zahavi (1975) proposed the so-called handicap principle to account for reliable signaling when interests diverge. At about the same time, Spence (1973) studied the same phenomenon for markets. The basic idea is simple: honest signaling can be maintained if the cost of signaling is high enough that using signals unreliably becomes too costly.

Zahavi's idea gained traction within biology after Grafen introduced a theoretical model underpinning the handicap principle Grafen (1990). Grafen's model focuses on sexual selection, but there are several other biological phenomena where the idea of a handicap can be applied, such as predator-prey signaling or child-parent communication. The Sir Philip Sidney game (Maynard Smith, 1991) represents the most well known example of the later. For this game theoretic model, and many others, costly signals enable honest communication in equilibrium.

Although the idea of costly signals is supported by both theoretical and empirical considerations, the costly signaling explanation also faces a number of problems. On the theoretical side, the costly signaling equilibrium may be very unattractive when compared to a pooling equilibrium where no information is transmitted (Bergstrom and Lachmann, 1997). Furthermore, dynamical models suggest that the signaling equilibrium is not easy to reach under several evolutionary dynamics (Huttegger and Zollman, 2010; Wagner, 2013). On the empirical side it has been difficult to measure signal costs that would be sufficiently high in order to support the costly signaling hypothesis (Searcy and Nowicky, 2005).

The empirical results suggest that communication might be adequately reliable even if the costs of signaling are small. Huttegger and Zollman (2010), Wagner (2013) and Zollman et al. (2013) have shown that in several costly signaling games there exists a so-called *hybrid equilibrium* in which signaling is not completely reliable but which has low signal costs.¹ In these papers, it has been demonstrated that the hybrid equilibrium is a Liapunov stable rest point of the replicator dynamics (Taylor and Jonker, 1978; Hofbauer and Sigmund, 1998). However, it is not asymptotically stable since there are two purely imaginary eigenvalues. This renders the replicator dynamics close to the hybrid equilibrium structurally unstable (Guckenheimer and Holmes, 1983): small perturbations of the replicator equations will result in a topologically different dynamic behavior. As a result, the stability of the equilibrium in these models might not translate into real cases where the actual dynamics diverges somewhat from the replicator dynamics.

The purpose of this paper is to study the robustness of the hybrid equilibrium. First, we consider a plausible perturbation of the replicator dynam-

¹ The order of the publications is somewhat misleading since Wagner was actually the first to recognize the significance of hybrid equilibria for evolutionary games.

ics that introduces mutation terms called the selection-mutation dynamics (Bürger, 1955; Hofbauer, 1985; Hofbauer and Sigmund, 1998).² It will be shown that under the selection-mutation dynamics rest points close to the hybrid equilibrium exist. We will characterize these perturbed rest points in terms of the mutation parameters. Furthermore, we will show that perturbed rest points are linearly stable, provided that mutation is sufficiently small. Second, we consider another version of the replicator dynamics known as the adjusted replicator dynamics (Maynard Smith, 1982). For this dynamics the hybrid equilibrium is not just Liapunov stable, but also asymptotically stable.

Our analysis is based on a simple costly signaling game that will be introduced in the next section. The results we obtain lend some support to the theoretical robustness of hybrid equilibria, reinforcing their importance as a potential explanatory factor in signaling interactions.

2 The model

2.1 Costly signaling games

The biological situation of central interest in costly signaling theory is one of partial conflict of interest. There is a “sender” that can condition its behavior on some private information. The “receiver” would benefit from conditioning its behavior on that information, but cannot do so directly. When the information takes on some values, the sender’s and receiver’s interests are aligned. But, when it takes on other values they are not.

In the context of sexual selection, this is modeled by presuming the sender is a male that is either of high or low quality. When the male is of high quality, both the male and the female benefit from mating. When the male is of low quality, the male benefits from mating but the female does better by declining. It is supposed that the male has at its disposal two phenotypes – signals – that can be displayed conditional on his quality and that can be observed by the female. The model allows the possibility that the phenotypes come at different costs to the male.

This situation is modeled by the game presented in figure 1 (from Zollman et al., 2013). If the cost of sending the signal is sufficiently high for the low quality male ($c_2 > 1$) and sufficiently low for the high quality male ($c_1 < 1$), then there is a Nash equilibrium where the male sends the signal only when the male is of high quality, and where the female only mates with males who send the signal (Zollman et al., 2013). This is an example of a classic costly signaling model.

However, when the cost for the low type is too low to sustain an honest signaling equilibrium, there nonetheless exists another equilibrium (the hybrid equilibrium) that features partial communication – the hybrid equilibrium. In this equilibrium the high quality sender always sends the signal, while the low

² In the context of signaling games, see Nowak (2000) and Nowak et al. (2002) for a general selection-mutation dynamics.

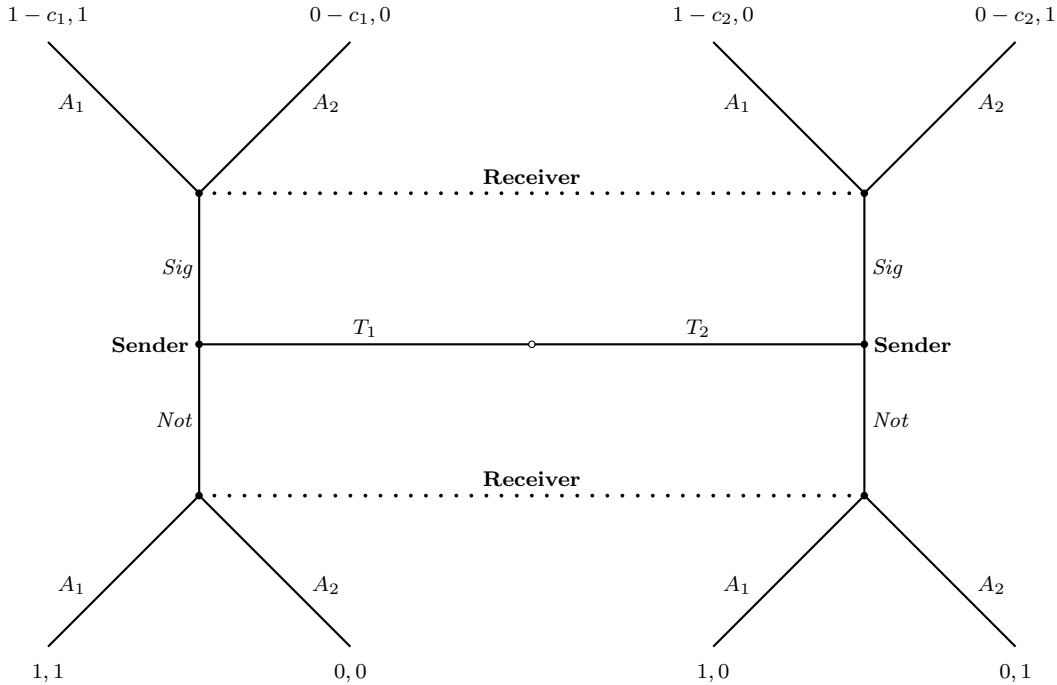


Fig. 1 Extensive form representation of a situation of communication between potential mates from Zollman et al. (2013). The first move is made by nature who chooses whether the male is of high quality (T_1) or low quality (T_2). The male can now condition its behavior on its quality, and can either send the signal or not. The cost to the signal varies with the quality of the male, c_1 if the male is of high quality and c_2 if of low quality. The female observes the males decision, but not the decision by nature, and chooses either to mate (A_1) or decline (A_2).

quality sender does so only occasionally. The receiver occasionally mates with a male who sends the signal, but always declines to mate with one who does not. In this equilibrium partial information is transferred.

Zollman et al. (2013) showed that in this equilibrium the low quality sender mixes with probability $m/(1 - m)$ where m is the probability of being of the high type. The female chooses to mate with a male who sends the signal with probability given by c_2 . This equilibria exists when $1 > c_2 > c_1 \geq 0$, precisely when the cost is too low to sustain the signaling equilibrium. Similar results have been shown for the Sir Philip Sidney game Huttegger and Zollman (2010) and for the Spence job market signaling game Wagner (2013).

2.2 Dynamics

That an equilibrium exists is insufficient to show that it will be the result of evolution by natural selection. In order to determine the evolutionary signifi-

cance of this equilibrium Zollman et al. (2013) considered the two-population replicator dynamics, which is given by:

$$\dot{x}_i = x_i(\pi_i(\mathbf{y}) - \pi(\mathbf{x}, \mathbf{y})) \quad (1a)$$

$$\dot{y}_j = y_j(\pi_j(\mathbf{x}) - \pi(\mathbf{y}, \mathbf{x})) \quad (1b)$$

Here, $\mathbf{x} = (x_1, \dots, x_N)$ is the state of the first population and $\mathbf{y} = (y_1, \dots, y_M)$ is the state of the second population. $\pi_k(\cdot)$ is the payoff to strategy k against the current population state of the other population, and $\pi(\cdot, \cdot)$ is the current average payoff in the population. In the costly signaling game described above $N = M = 4$. Thus the state space of the selection-mutation dynamics is a sixteen-dimensional polytope.

The replicator dynamics of this game was studied in Zollman et al. (2013). There, it is shown that the hybrid equilibrium is Liapunov stable (provided that it exists). More specifically, it is a spiraling center. It has four negative transversal eigenvalues, and two purely imaginary eigenvalues. This implies that, from the interior, trajectories close to the hybrid equilibrium converge to the plane that supports the hybrid equilibrium. On the plane, solution trajectories cycle around the hybrid equilibrium.

This has two important consequences. First, the hybrid equilibrium is Liapunov stable but not asymptotically stable. Second, the dynamics close to the hybrid equilibrium is not structurally stable (Guckenheimer and Holmes, 1983). This means that small perturbations of the replicator equations (1) will result in qualitatively different dynamical behavior close to the hybrid equilibrium.

This raises the question whether the hybrid equilibrium continues to be dynamically stable under biologically plausible perturbations. One such perturbation is given by supplementing the replicator dynamics with mutation terms. This is achieved by the selection-mutation dynamics which uses deterministic mutation terms (Hofbauer, 1985). The selection-mutation dynamics for signaling games without costs was studied in Hofbauer and Huttegger (2008).

In the simplest case of uniform mutation rates, the selection-mutation dynamics is given by:

$$\dot{x}_i = x_i(\pi_i(\mathbf{y}) - \pi(\mathbf{x}, \mathbf{y})) + \varepsilon(1 - Nx_i) \quad (2a)$$

$$\dot{y}_j = y_j(\pi_j(\mathbf{x}) - \pi(\mathbf{y}, \mathbf{x})) + \delta(1 - My_j) \quad (2b)$$

ε and δ are the mutation rates in the first and the second population, respectively. We allow the mutation rates to be different, but require that they are of the same order; i.e., $\varepsilon = \rho\delta$ for some constant ρ as $\varepsilon \rightarrow 0$. This implies, in particular, that $O(\varepsilon^k) = O(\delta^k)$ for all $k \geq 1$. In what follows $O(2)$ denotes terms of the form $\varepsilon^2, \delta^2, \varepsilon\delta$ or similar terms of higher order.

The terms $\varepsilon(1 - Nx_i)$ and $\delta(1 - My_j)$ represent the effects of mutation. Notice that, if $\varepsilon, \delta = 0$, (2) reduces to the standard replicator dynamics. If

$\varepsilon, \delta > 0$, then there are no rest points at the boundary of the state space. Hence, the effect of the mutation terms is to introduce strategies that are not present in the population.

Suppose that the signaling equilibrium exists. Then it is a hyperbolic rest point of the replicator dynamics (1). Hence, the Jacobian of the replicator dynamics (1) evaluated at the signaling equilibrium is invertible. Thus, by the implicit function theorem, there exists a unique rest point of the selection-mutation dynamics close to the signaling equilibrium for sufficiently small ε, δ . Since the eigenvalues of the selection-mutation dynamics are continuous in ε, δ , this implies the following proposition:

Proposition 1 *If (c_1, c_2, m) are such that the signaling equilibrium exists, then there is a unique linearly stable rest point of the selection-mutation dynamics (2) close to the signaling equilibrium for sufficiently small ε, δ .*

As a consequence, the stability of the signaling equilibrium is robust under the selection-mutation dynamics. In the next section we study the question of whether something similar holds for the hybrid equilibrium.

3 Results

3.1 Existence and uniqueness of perturbed rest points

We note first that the hybrid equilibrium is *regular*; that is, the Jacobian matrix of the replicator dynamics evaluated at the point $(\mathbf{x}^*, \mathbf{y}^*) = [(1 - \alpha, \alpha, 0, 0), (c_2, 1 - c_2, 0, 0)]$ is a regular matrix (and thus invertible). (Here $\alpha = m/(1 - m)$.) The Jacobian is regular at the hybrid equilibrium since it has no zero eigenvalue. As noted above, it has four negative eigenvalues and two non-zero, purely imaginary eigenvalues.

Since the hybrid equilibrium is regular, it follows from the implicit function theorem that there exists a unique rest point $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ of (2) close to the hybrid equilibrium $(\mathbf{x}^*, \mathbf{y}^*)$, provided that ε, δ are sufficiently small. The location of $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ depends on ε, δ and the other parameters of the game. The following result makes this more precise.

Proposition 2 *If (c_1, c_2, m) are such that the hybrid equilibrium exists, then there is a unique rest point $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ close to the hybrid equilibrium for sufficiently small ε, δ . The perturbed rest point $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ is given by*

$$\bar{x}_3 = \bar{x}_4 = \frac{\varepsilon}{(c_2 - c_1)m} + O(\varepsilon^2) \quad \text{and} \quad \bar{y}_3 = \bar{y}_4 = \frac{\delta}{1 - 2m} + O(\delta^2)$$

and

$$\bar{x}_1 = \frac{1 - 2m}{1 - m} - \frac{\varepsilon(3m - 1)}{m(1 - m)(c_2 - c_1)} - \frac{\delta(1 - 2c_2)}{(1 - m)c_2(1 - c_2)} + O(2)$$

$$\bar{y}_2 = c_2 - \frac{\varepsilon(3 - 7m)}{m(1 - 2m)} + \frac{\delta}{1 - 2m} + O(2).$$

Proof Since the hybrid equilibrium $(\mathbf{x}^*, \mathbf{y}^*)$ is regular, we know that the perturbed rest point $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ exists close to the it for sufficiently small ε, δ . We can thus study Taylor expansions of the equations (2) around the hybrid equilibrium $(\mathbf{x}^*, \mathbf{y}^*)$ in order to determine its location.

Let

$$f_i(\mathbf{x}, \mathbf{y}) = x_i(\pi_i(\mathbf{y}) - \pi(\mathbf{x}, \mathbf{y})) + \varepsilon(1 - 4x_i)$$

and

$$g_j(\mathbf{x}, \mathbf{y}) = y_j(\pi_j(\mathbf{x}) - \pi(\mathbf{y}, \mathbf{x})) + \delta(1 - 4y_j).$$

for $1 \leq i, j \leq 4$. We start by considering x_3 . Here we have

$$\frac{\partial f_3}{\partial x_k} = 0$$

for all $k \neq 3$ and

$$\frac{\partial f_3}{\partial y_k} = 0$$

for $1 \leq k \leq 4$. Since $f_3(\mathbf{x}^*, \mathbf{y}^*) = \varepsilon$ and $\partial f_3(\mathbf{x}^*, \mathbf{y}^*)/\partial x_3 = -4\varepsilon + (c_2 - c_1)m$, the Taylor expansion of f_3 around the hybrid equilibrium $(\mathbf{x}^*, \mathbf{y}^*)$ is given by

$$f_3(\mathbf{x}, \mathbf{y}) = \varepsilon + (-4\varepsilon + (c_1 - c_2)m)x_3 + O(x_3^2).$$

Setting $f_3(\mathbf{x}, \mathbf{y}) = 0$ yields the solution

$$\bar{x}_3 = \frac{\varepsilon}{(c_2 - c_1)m} + O(\varepsilon^2).$$

The argument can be repeated for \bar{x}_4 . Similarly, $\partial g_3/\partial y_3$ is the only non-zero partial derivative of g_3 . Thus the Taylor expansion of g_3 around the hybrid equilibrium is given by

$$g_1(\mathbf{x}, \mathbf{y}) = \delta + (-1 - 4\delta + 2m)y_3 + O(y_3^2)$$

for (\mathbf{x}, \mathbf{y}) close to $(\mathbf{x}^*, \mathbf{y}^*)$. The same holds for y_4 . This yields the solutions \bar{y}_1, \bar{y}_2 .

We consider the Taylor expansion for f_1 next. Note first that

$$f_1(\mathbf{x}^*, \mathbf{y}^*) = -\frac{\varepsilon(3 - 7m)}{1 - m}.$$

Calculating the partial derivatives of f_1 relative to all variables, we have

$$\begin{aligned} f_1(\mathbf{x}, \mathbf{y}) = & -\frac{\varepsilon(3 - 7m)}{1 - m} - 4\varepsilon \left(x_1 - \frac{1 - 2m}{1 - m} \right) + \frac{(c_2 - c_1)m(1 - 2m)}{1 - m}x_3 - \frac{(c_2 - c_1)m(1 - 2m)}{1 - m}x_4 \\ & + \frac{m(2m - 1)}{1 - m}(y_1 - c_2) + \left(1 - \frac{1}{1 - m} + 2m \right) y_4 + \text{higher order terms} \end{aligned}$$

close to $(\mathbf{x}^*, \mathbf{y}^*)$. Furthermore,

$$g_1(\mathbf{x}^*, \mathbf{y}^*) = \delta - 4c_2\delta$$

and

$$g_1(\mathbf{x}, \mathbf{y}) = \delta - 4c_2\delta + (1 - c_2)c_2(1 - m) \left(x_1 - \frac{1 - 2m}{1 - m} \right) + (1 - c_2)c_2(1 - 2m)x_3 - (1 - c_2)c_2mx_4 \\ - 4\delta(y_1 - c_2) + (c_2 - 2c_2m)(y_3 + y_4) + \text{higher order terms}$$

close to $(\mathbf{x}^*, \mathbf{y}^*)$. If we use the solutions obtained for $\bar{x}_3, \bar{x}_4, \bar{y}_3, \bar{y}_4$, then the expansions for f_1 and g_1 yield a system of two equations in x_1 and y_1 :

$$0 = \frac{1}{1 - m} ((\varepsilon(1 - m - 4x_1 + 4mx_1) - m(-\delta - c_2(1 - 2m) + y_1 - 2my_1))) \\ 0 = \frac{(1 - c_2)c_2\varepsilon}{(c_1 - c_2)} + \delta - 2c_2\delta + \frac{(1 - c_2)c_2\varepsilon(1 - 2m)}{(c_2 - c_1)m} - (1 - c_2)c_2(1 - m(2 - x_1) - x_1) + 4\delta(c_2 - y_1).$$

\bar{x}_1 and \bar{y}_2 are the solutions of this system.

Proposition 2 should be contrasted with the results obtained in Hofbauer and Huttegger (2008), where the selection-mutation dynamics is applied to signaling games without conflicts of interests and costs. In these games, perturbed rest points that are not close to a signaling equilibrium can vary in terms of their number depending on the ratio ε/δ . This is not the case here because the hybrid equilibrium is regular.

3.2 Dynamic stability of perturbed rest points

Next, we study the stability properties of the rest point $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$. Let us first fix the parameters c_1, c_2 and m . Since c_1 is the low-cost signal, we let $c_1 = 0$. Moreover, we set $c_2 = \frac{1}{2}$ and $m = \frac{1}{3}$. In this case, the location of the hybrid equilibrium $(\mathbf{x}^*, \mathbf{y}^*)$ is

$$\left(\frac{1}{2}, \frac{1}{2}, 0, 0; \frac{1}{2}, \frac{1}{2}, 0, 0 \right).$$

The eigenvalues of the replicator dynamics evaluated at the hybrid equilibrium are

$$-\frac{1}{6}, -\frac{1}{6}, -\frac{1}{3}, -\frac{1}{3}, -\frac{i}{6}, \frac{i}{6}.$$

Proposition 2 determines the location of the rest point $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ close to the hybrid equilibrium. In particular,

$$\bar{x}_3 = \bar{x}_4 = 6\varepsilon + O(\varepsilon^2), \quad \bar{y}_3 = y_4 = 3\delta + O(\delta^2) \quad (3)$$

and

$$\bar{x}_1 = \frac{1}{2} + O(2), \quad \bar{y}_1 = \frac{1}{2} - 6\varepsilon + 3\delta + O(2). \quad (4)$$

With these values, we can now demonstrate that the perturbed rest point is hyperbolic and asymptotically stable.

Lemma 1 *The perturbed rest point of Proposition 2 with the parameter values $c_1 = 0, c_2 = \frac{1}{2}, m = \frac{1}{3}$ is linearly stable under the selection-mutation dynamics (2) for sufficiently small ε, δ .*

Proof At the hybrid equilibrium (\bar{x}, \bar{y}) with values given in (3) and (4) the Jacobian matrix is given by (up to higher order terms in ε, δ):

$$J = \begin{pmatrix} -2\varepsilon & \frac{1}{12} - 3\varepsilon & \frac{1}{12} - \varepsilon - 2\delta & -\frac{1}{6} + 4\varepsilon & -2\varepsilon & \frac{1}{6} - 6\varepsilon \\ 0 & -\frac{1}{6} + 3\varepsilon & \varepsilon & -4\varepsilon & 0 & 4\varepsilon \\ 0 & \varepsilon & -\frac{1}{6} - \varepsilon + 4\delta & 0 & 4\varepsilon & 4\varepsilon \\ \frac{1}{6} + \delta & \frac{1}{12} + \frac{\delta}{2} & -\frac{1}{12} - \frac{\delta}{2} & -2\delta & \frac{1}{6} - 2\varepsilon + \delta & \frac{1}{6} - 2\varepsilon + \delta \\ -\delta & -\frac{\delta}{2} & \frac{\delta}{2} & 0 & -\frac{1}{3} - \delta & \delta \\ -3\delta & -\frac{3\delta}{2} & \frac{3\delta}{2} & 0 & \delta & -\frac{1}{3} - \delta \end{pmatrix}$$

Let $\chi(J)$ be the characteristic polynomial of J . We may only consider the linear approximation for $\chi(J)$ and ignore higher-order terms in ε and δ . The zero-order term of $\chi(J)$ developed in z is equal to

$$\frac{1}{11664} + \frac{z}{648} + \frac{17z^2}{1296} + \frac{z^3}{12} + \frac{7z^4}{18} + z^5 + z^6.$$

The terms linear in ε are given by

$$-\varepsilon \left(\frac{1}{486} + \frac{2z}{81} + \frac{5z^2}{54} + \frac{z^3}{9} \right).$$

The terms linear in δ are given by

$$\delta \left(-\frac{1}{486} - \frac{z}{81} + \frac{z^2}{18} + \frac{4z^3}{9} + \frac{2z^4}{3} \right).$$

The sum of these three expressions is the product of four factors (multiplied by the factor $-1/11664$). The first three factors are

$$(1 + 3x)^2 \quad \text{and} \quad 1 + 6x.$$

Setting these two factors to zero yields the eigenvalues

$$-\frac{1}{3} + O(2), \quad -\frac{1}{6} + O(2)$$

If ε, δ are small, these eigenvalues are negative.

The third factor is a polynomial of degree three. It is given by

$$-1 + 24\varepsilon + 24\delta - 6z - 144\delta z - 36z^2 - 216z^3. \quad (5)$$

The corresponding cubic equation can be solved. The three solutions can be developed into Taylor expansions around $(\varepsilon, \delta) = (0, 0)$. This yields one real eigenvalue

$$-\frac{1}{6} + 2\varepsilon + 4\delta + O(2).$$

For small ε, δ , this eigenvalue is negative.

The polynomial (5) has two more solutions, which are complex conjugate. By again using Taylor expansions around $(\varepsilon, \delta) = (0, 0)$ they can be given as

$$-(1-i)\varepsilon - 2\delta - \frac{i}{6} + O(2)$$

and

$$-(1+i)\varepsilon - 2\delta + \frac{i}{6} + O(2).$$

The real parts of the two eigenvalues are negative for $\varepsilon, \delta > 0$. Together with the foregoing results, this shows that $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ is linearly stable for sufficiently small ε, δ .

Lemma 1 shows that the hybrid equilibrium is evolutionarily significant from the point of view of the selection-mutation dynamics for the specific parameter values chosen. The same calculations can be performed for other values of the parameters. For instance, choosing $c_1 = 0, c_2 = 1/4$ and $m = 1/3$ also leads to the conclusions of Lemma 1. Similar calculations can be used in the case of other costly signaling games. For instance, an analogue of the foregoing lemma is true for the Sir Philip Sidney game.

Lemma 1 leads straightforwardly to our main result.

Theorem 1 *There is an open set of parameter values (c_1, c_2, m) such that the perturbed rest points as in Proposition 2 exist and are linearly stable under the selection-mutation dynamics (2) for sufficiently small ε, δ .*

Proof The system (2) is continuous in the parameters (c_1, c_2, m) . It follows that also the Jacobian and the characteristic polynomial are continuous in c_1, c_2, m , and thus the same is true for the eigenvalues. Since, by Lemma 1, the perturbed rest point of Proposition 2 is linearly stable for $(c_1, c_2, m) = (0, 1/2, 1/3)$, the eigenvalues will be close to the eigenvalues of this rest point as long as (c_1, c_2, m) is close to $(0, 1/2, 1/3)$. Because the hybrid equilibrium exists for these values, the assertion of the theorem follows.

Theorem 1 shows that the result obtained in Lemma 1 can be robustly extended to other values of the model parameters.

3.3 Simulations

The analytic results of the previous section have established the existence of open sets of games where the hybrid equilibrium is asymptotically stable for sufficiently small mutations. We cannot, however, analytically determine in what proportion of signaling games with a hybrid equilibrium the equilibrium is asymptotically stable. For this reason we turn to numerical techniques.

To do so, we randomly generated parameterizations of the game from figure 1 which featured the hybrid equilibrium. Utilizing Mathematica, we calculated the eigenvalues of the Jacobian at the point given in 2 for two different

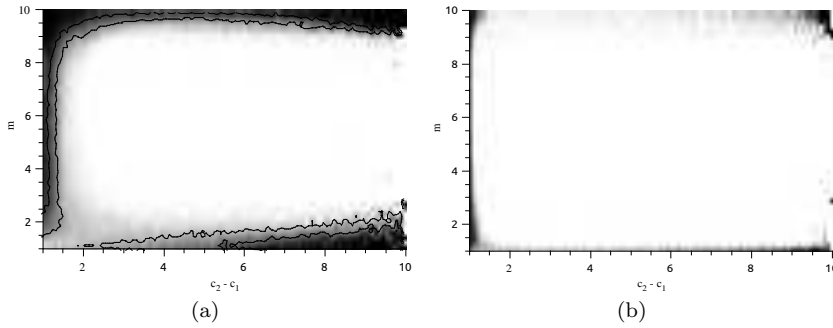


Fig. 2 Numerical results for the stability of the hybrid equilibrium. The x-axis represents the difference in cost between the high and low cost signals. The y-axis represents the probability that the sender is of type T_1 . Black indicates that all generated games at that point featured at least one positive eigenvalue. White indicates that no generated games at that point featured at least one positive eigenvalue.

mutation parameters $\epsilon, \delta = 0.01$ and 0.001 . For the larger mutation parameter we generated 499,351 sample points, and for the smaller we generated 499,222 points. We determined whether any of the eigenvalues of the Jacobian at the hybrid equilibrium were positive. The results of this numerical analysis are illustrated in figure 2.

We can conclude several things. First, the hybrid equilibrium is not stable in the replicator-mutator dynamics in all games that have a hybrid equilibrium. Second, while there were many games where the rest point was unstable, almost all of them lay at the edges of the parameter space – that is near the boundary where the hybrid equilibrium no longer exists. For a large number of points in the interior of the parameter space (which includes our example from the last section), the hybrid equilibrium is stable.

Finally, as the mutation parameter becomes smaller, it appears that the region where the hybrid equilibrium is stable grows. This suggests a conjecture: that as $\delta, \epsilon \rightarrow 0$, the proportion of the parameter space where the hybrid equilibrium is stable (among those areas where it exists) goes to 1.

4 The adjusted replicator dynamics

Another robustness result can be straightforwardly obtained by considering the *adjusted replicator dynamics* that was introduced by Maynard Smith (1982). In the two-population context, this dynamics is given by:

$$\dot{x}_i = \frac{x_i(\pi_i(\mathbf{y}) - \pi(\mathbf{x}, \mathbf{y}))}{\pi(\mathbf{x}, \mathbf{y})} \quad (6a)$$

$$\dot{y}_j = \frac{y_j(\pi_j(\mathbf{x}) - \pi(\mathbf{y}, \mathbf{x}))}{\pi(\mathbf{y}, \mathbf{x})} \quad (6b)$$

Thus, per capita growth rates are normalized by the population's average fitness.

It is not difficult to see that the hybrid equilibrium of the game in figure 1 is asymptotically stable for the dynamics (6). First, note that the transversal eigenvalues of the Jacobian of (6) evaluated at the hybrid equilibrium must have the same sign as the transversal eigenvalues of the standard replicator dynamics (1) since the adjusted replicator dynamics is a normalization of the standard replicator dynamics. Hence, all transversal eigenvalues are real and negative.

Second, consider the plane on which the hybrid equilibrium lies (i.e., the boundary spanned by its support). The signaling game restricted to the support of the hybrid equilibrium is a version of Matching Pennies. It is well known that the adjusted replicator dynamics (6) converges to the unique Nash equilibrium of games like Matching Pennies, for the eigenvalues of the Nash equilibrium have negative real part (Weibull, 1995, 176–179). Overall it follows that all eigenvalues of the hybrid equilibrium have negative real part.

5 Conclusion

After the publication of Huttegger and Zollman (2010) and Zollman et al. (2013) it remained a possibility that the hybrid equilibrium was non-generic. That is, the reported results did not exclude the possibility that the hybrid equilibrium would be rendered unstable if small perturbations to the replicator dynamics, like those from mutation, were introduced.

In this paper we establish a number of results that suggest that the stability of hybrid equilibrium is generic, and that as mutation is introduced the equilibrium actually becomes *more* significant. We illustrated that the hybrid equilibrium remains a rest point and there is an open set of parameter values where it is asymptotically stable. Furthermore, we established (through numerical methods) that this constitutes a relatively large portion of the parameter space, and that it grows as the mutation parameter shrinks.

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