Particle Diffusion from the Beam-Beam Interaction in Synchrotron Colliders

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We investigate the beam-beam interaction in a synchrotron collider, specifically studying slow particle diffusion in phase space away from tune resonances. Using the tune and tune shift of contemporary large hadron colliders as reference parameters, our computation shows all particles diffusive after 10^5 rotations in contrast to previous single particle tracking results. The diffusion coefficients are several orders of magnitude higher than the tracking code and increase exponentially with the action, caused by the collision induced variation of the second moment of the beams $\langle x^2 \rangle$.

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In synchrotron colliders one of the principal limitations on beam intensity is the beam-beam interaction [1,2]. In the beam-beam interaction each beam imparts an impulse on the other beam at the interaction point (IP) where the beams cross. For the hadron colliders the beam-beam interaction is expected to be crucial, since there is little synchrotron radiation damping to slow beam emittance growth as in electron storage rings [1].

In this paper we investigate the beam-beam interaction with emphasis on subtle particle diffusion away from resonances on time scales of the order of machine operation times ($\sim 10^4$ s). A one dimensional model is employed at the IP so that oscillations in only one transverse direction due to the counterstreaming beams are studied. The rest of the machine is treated by symplectic harmonic transport (betatron oscillations). By employing a fully self-consistent model at the interaction point, an assessment of the relative importance of collisions as a whole and individual "soft" collisions (collective effects) can be determined. Specifically, we will examine the contribution of self-consistent effects on particle diffusion after a large number of interactions.

We briefly describe the conventional tracking and our new δf codes used to study the beam-beam interaction. More detailed descriptions of the δf code can be found in other references [3–6]. The basic principle of conventional tracking codes is to follow the dynamics of single particles around the machine [7,8]. In the beam-beam interaction the tracking code consists of two components: a target beam and a projectile beam. The target beam is treated as a rigid smooth Gaussian distribution of a large number of particles. It remains unchanged from interaction to interaction. The projectile beam is considered to be a collection of mutually noninteracting particles which are perturbed by the target beam. This is the so-called weak-strong approximation [7,8]. In tracking code simulations in the weak-strong approximation, transport about one turn is simulated as the product of two matrices, one for the one turn Courant-Synder map [9], and the other for the impulsive application of the beam-beam interaction discussed above [7,8]:

$$\begin{bmatrix} x \\ x' \end{bmatrix}_{\text{final}} = \begin{bmatrix} \cos(2\pi\nu_0) & \beta_0^* \sin(2\pi\nu_0) \\ -\sin(2\pi\nu_0)/\beta_0^* & \cos(2\pi\nu_0) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 4\pi\Delta\nu_0 F(x)/\beta_0^* & 1 \end{bmatrix} \begin{bmatrix} x \\ x' \end{bmatrix}_{\text{initial}}, \tag{1}$$

where x is the position of the particle, x' is dx/ds, s is the distance along the collider, $\nu_0 = \oint ds/\beta(s)$ is the tune, $\Delta\nu_0$ is the input tune shift, β_0^* is the betatron oscillation amplitude at the IP, and $F(x)$ is the force of a 1D Gaussian slab:

$$F(x) = \sqrt{\frac{\pi}{2}} \left(\frac{\sigma_{x0}}{x}\right) \operatorname{erf}\left(\frac{x}{\sqrt{2}\sigma_{x0}}\right), \tag{2}$$

where erf is the error function and σ_{x0} is the beam standard deviation in x.

In a fully self-consistent "strong-strong" treatment the force $F(x)$ in Eq. (2) is solved for both beams using the Vlasov equation for the beam-beam interaction [10]:

$$\frac{\partial f}{\partial s} + x' \frac{\partial f}{\partial x} - [K(s)x - F(x)\delta_p(s)] \frac{\partial f}{\partial x'} = 0, \tag{3}$$

where $K(s)x$ is the magnetic guiding force, $F(x)$ is the beam kick from the other beam, and $\delta_p(s)$ is the periodic delta function which is nonzero at only the IP.

In previous strong-strong treatments the distribution f was represented either by a Gaussian with varying position and width [11] or by a finite number of particles [12]. Each method has its advantages and disadvantages. While the advantage of the Gaussian treatment is the lack of finite particle fluctuation noise, the disadvantage is that higher moments of the distribution do not evolve. The finite particle method allows the evolution of the higher moments, but is subject to the noise.

The δf method substantially reduces the fluctuation noise and allows the evolution of higher moments by evolving only the perturbative part of the distribution [3–5]. The total distribution function $f(x, x', s)$ is decomposed into

$$f(x, x', s) = f_0(x, x', s) + \delta f(x, x', s), \tag{4}$$

where $f_0(x, x', s)$ is the steady or slowly varying part of

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the distribution and $\delta f(x,x',s)$ is the perturbative part. For beam-beam interaction an analytic solution close to the original Vlasov equation can be found using a linearized beam-beam force whose solution is a Gaussian distribution. Equation (3) reduces to an exact form:

$$\frac{\partial \delta f}{\partial s} + x'\frac{\partial \delta f}{\partial x} - [K(s)x - F_0(x)\delta_p(s)]\frac{\partial \delta f}{\partial x'} = -[F(x) - F_0(x)\delta_p(s)]\frac{\partial f_0}{\partial x'}, \quad (5)$$

where $F_0(x)$ is the kick from a Gaussian beam, $F(x)$ is the kick from a Gaussian beam $F_0(x)$ plus $\delta F(x)$ the force due to the perturbed distribution δf, and F_0 is the portion of the beam-beam force $F(x)$ linear in x. Only the nonlinear part of the beam-beam force on the right hand side of Eq. (5) is used to advance δf. The perturbative part $\delta f(x,x',s)$ is then small, causing only small changes to the distribution representing the fluctuation levels. Equation (5) can be represented by N particles by the method of characteristics:

$$\delta f(x,x',s) = \sum_{i=1}^{N} w_i[s, x_i(s), x_i'(s)]\delta(x - x_i(s))\delta(x' - x_i'(s)), \quad (6)$$

where the weights of particles w_i are evolved using Eq. (5).

We examine particle diffusion brought about by the self-consistent solution of the beam-beam interaction through the δf simulation that goes beyond the results from the weak-strong tracking code. We also determine the contribution from beam moments to the diffusion.

Figure 1(a) shows the diffusion coefficients of 100 randomly distributed sample particles versus their initial action calculated for the δf code and the tracking code, where the normalized action is $J = \frac{1}{2}[(\frac{x}{\sigma_x})^2 + (\frac{p_x}{\sigma_p})^2]$. The diffusion coefficients are obtained after 10^5 rotations with the tune $\nu_0 = 0.285$ and tune shift $\Delta\nu_0 = 2.1\times10^{-3}$ similar to large hadron collider reference parameters. The diffusion coefficient $D(J)$ in the action is normalized to σ_J^2/N_r, where $\sigma_J = \sigma_x^2/\beta^*$ and N_r is the number of rotations so that it takes $D(J)^{-1}$ turns to diffuse over one standard deviation of the beam emittance. For example, Fig. 1(a) shows on average that the diffusion time is approximately 10^{10} turns in our self-consistent calculations. The distinction between the real diffusion and apparent diffusion such as phase space oscillations is determined using the method of Chirikov where coefficients D_1 and D_2 are calculated over two different time scales [1,2]. If $D_1 \approx D_2$, then the motion in J is diffusive. If $D_1 \gg D_2$ where D_1 is calculated on shorter time scales than D_2, then the motion is merely phase space oscillations. In Fig. 1(a) the ratio of the coefficients calculated over intervals of 10^3 and 10^4 turns for the tracking code sample particles is on the order of $\frac{D_2}{D_1} \sim 0.01 \to 0.1$ which indicates little diffusive behavior.

We find (i) that the $D(J)$ is far greater in the δf code than in the tracking code, indicating that conventional tracking code prediction is unrealistically low; (ii) that even the highest value of $D(J)$ from the self-consistent result remains within typical machine design lifetimes of 10^8 turns; and (iii) there appears a strong action, J, dependence. In Fig. 1(a) for all sample particles in the δf code the ratio D_2/D_1 is on the order of 1, indicating that all particles are diffusive. The diffusion coefficient $D(J)$ is an approximately exponential function of J for $J > 0.5$. We also find that the alternate breathing oscillations of the two beams appear in the self-consistent calculations, but not in the tracking calculations. The onset of the oscillations is due to the collective interaction. The diffusion is not strongly dependent on the number of simulation particles. In this δf simulation each beam has 10^3 simulation particles; however, the coefficients calculated for a simulation with 10^4 simulation particles give the same results [6].

The source of the enhanced diffusion in the self-consistent δf simulation is identified with the observed variation of the moments of each beam which does not occur in the tracking code. The contribution of the first two beam moments $\langle x\rangle$ and $\langle x^2\rangle$ to the beam diffusion may be estimated by varying these moments in the tracking code which assumes a Gaussian beam.

When the beam moment $\langle x\rangle$ from the δf code is input into the tracking code, the diffusion coefficients calculated for sample particles with $J < 1$ are close to that of the δf code. However, for $J > 1$ the diffusion coefficients level off and deviate substantially from the exponentially

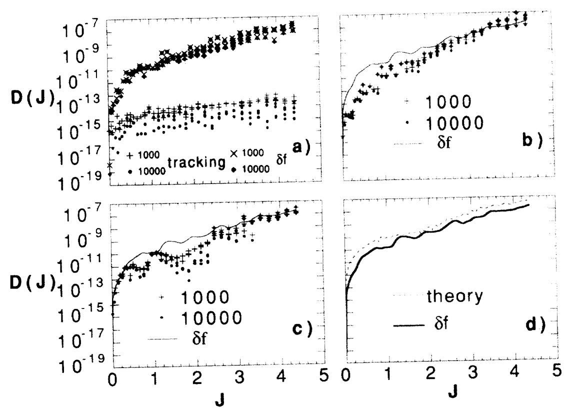

FIG. 1. $D(J)$ from (a) our δf code and conventional tracking code, (b) tracking code particles with variations in $\langle x^2\rangle$ input from the δf, and (c) tracking code particles with $\langle x^2\rangle$ input using a band of frequencies around $2(\nu_0-\Delta\nu)$ for $M = 10^5$ rotations. D_1 and D_2 have time scales of $\Delta N_1 = 1000$ and $\Delta N_2 = 10\,000$ rotations, respectively. (d) The diffusion $D(J)$ from δf and our analytic theory.

increasing diffusion coefficients of the δf code. Figure 1(b) shows diffusion coefficients from the tracking code particles when the beam σ_x of the tracking code is varied using $\langle x^2 \rangle$ from the δf code. The solid curve for the δf diffusion is obtained by smoothing the diffusion coefficients calculated for 10 000 rotations. The diffusion coefficients from the tracking code with the appropriate spectrum of variations of $\langle x^2 \rangle$ and the δf code nearly overlap for most values of $J > 2$. For values of $J < 2$ the tracking code coefficients are smaller than the δf code. Thus, most of the enhanced diffusion can be accounted for by the variation of the second moment $\langle x^2 \rangle$ incurred by collective "breathing" modes. Diffusion at the core of the beam can be accounted for by the variations in both the first $\langle x \rangle$ and second $\langle x^2 \rangle$ moments.

Figure 2(a) shows a portion of the frequency spectra $S_J(f)$ of four different sample particles initially at $J = 1, 2, 3$, and 4 over $M = 10^5$ rotations where

$$S_J(f) = \mathrm{FFT}[C(r)W(r)]. \tag{7}$$

$C(r)$ is the correlation function which is calculated from a discrete set of values of the action J for each particle [13]:

$$C(r) = \frac{1}{M-r} \sum_{n=1}^{M-r} J(n) J^*(n+r), \tag{8}$$

where $r = 0, ..., m$, r is the rotation lag, m is the maximum rotation lag, and M is the total number of rotations. $W(r)$ is a window function and FFT is a fast Fourier transform with M rounded to the nearest power of 2. The frequency f of the peak in $S_J(f)$ is decreasing with increasing initial J of each sample particle and corresponds approximately to $f = 1 - 2[\nu_0 - \Delta\nu(J)]$ where $\Delta\nu(J)$ is the tune shift of the particular particle. The decrease in frequency can be attributed to the decrease in $\Delta\nu(J)$ with increasing J of the particle, typical of the beam-beam tune shift. Figure 2(b) shows a portion of the frequency spectrum $S_{\sigma_x}(f)$ where the second moment of motion $\sigma_x = \sqrt{\langle x^2 \rangle}$ for $M = 10^5$ rotations is used in Eq. (8). The arrows indicate the upper and lower bounds of frequencies accessible to particles in the beam. The upper and lower bounds correspond to particles at the center of the beam experiencing the full tune shift $\Delta\nu_0$ and particles with $J \to \infty$ experiencing no tune shift, respectively. The frequency f of the peak in Fig. 2(b) of the σ_x motion is approximately $1 - 2(\nu_0 - \Delta\nu)$ where $\Delta\nu$ is the tune shift of the large J particles. Sample particles with large J are in resonance with the σ_x variation. Sample particles with small J have a characteristic frequency f in their motion which is higher than the σ_x frequency and are not in resonance. Therefore, the main contribution to the diffusion of the large J particles is resonance overlap [1].

Figure 1(c) shows the diffusion coefficients obtained from the input of $\langle x^2 \rangle$ variation into the tracking code including only the band of frequencies f shown in Fig. 2(b). There are some particles in the range $1 < J < 2.2$ whose diffusion coefficients are lower than that obtained from the δf simulation. However, the diffusion observed in the δf code can be mostly accounted for by the variation of $\langle x^2 \rangle$ in a narrow band of frequencies near $1 - 2(\nu_0 - \Delta\nu)$ where $\Delta\nu$ is that for large amplitude particles.

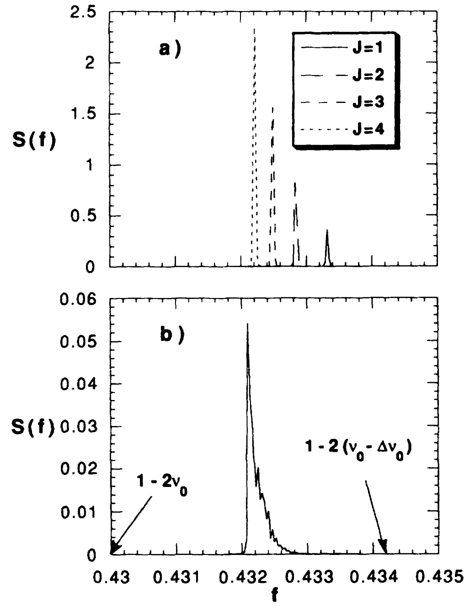

FIG. 2. After $M = 10^5$ rotations (a) part of the frequency spectra of the variation of the action J of four different sample particles from the tracking code with initial values of $J = 1, 2, 3$, and 4 and (b) part of the spectra of the variation of σ_x from the δf simulation where the arrows indicate the range frequencies possible for particles in the beam. The upper and lower bounds of frequencies correspond to particles with $J = 0$ ($\Delta\nu = \Delta\nu_0$) and $J \to \infty$ ($\Delta\nu = 0$), respectively.

The variation in $\langle x \rangle$ does not contribute as significantly to the particle diffusion in J as $\langle x^2 \rangle$. The characteristic frequencies of the $\langle x \rangle$ motion is not as close to the characteristic frequencies of the J variation as the $\langle x^2 \rangle$ motion.

An analytic expression for the diffusion in action J can be obtained for beam σ_x variation by adapting the formalism of Stupakov [14] which contained external kicks with $\langle x \rangle$ variation, but no $\langle x^2 \rangle$ variation. The change in the action due to the beam-beam kick from a one dimensional Gaussian slab can be written in the form

$$\Delta J = (2\pi)^{3/2} \Delta\nu_0 \sigma_x x' \mathrm{erf}\left(\frac{x}{\sqrt{2}\sigma_x}\right), \tag{9}$$

where

$$x = \sqrt{2J\beta}\cos(\Psi), \quad x' = -\sqrt{\frac{2J}{\beta}}\sin(\Psi), \tag{10}$$

and Ψ is the phase advance. Perturbing Eq. (9) with respect to σ_x and summing over M turns one obtains an expression for the change in the action J :

$$\Delta J_M = 8\pi\Delta\nu_0 \sum_{l=0}^{M-1} J_l \exp\left(-\frac{J_l\beta}{2\sigma_{x0}^2}\right) \sum_{k=0}^{\infty} \{I_k - I_{k+2}\}(-1)^k \sin[2(k+1)\Psi_l] \frac{\Delta\sigma_x(l)}{\sigma_{x0}}, \tag{11}$$

where I_k represents the kth modified Bessel function with arguments $(\frac{J_l\beta}{2\sigma_{x0}^2})$. The phase advance is $\Psi_l = 2\pi l[\nu_0 + \langle\Delta\nu(J)\rangle]$ where $\langle\Delta\nu(J)\rangle$ is the average tune shift that the particle encounters.

The diffusion coefficients can be calculated from

$$D(J) = \frac{\langle\Delta J_M^2\rangle}{M}, \tag{12}$$

where

$$\Delta J_M^2 = 32\pi^2\Delta\nu_0^2 J^2 \exp\left(-\frac{J\beta}{\sigma_{x0}^2}\right) \sum_{n=-\infty}^{\infty} K(n) \sum_{k=0}^{\infty} (I_k - I_{k+2})^2 \cos[2(k+1)\Psi_n]. \tag{13}$$

Equation (13) is obtained by squaring Eq. (11) and assuming that the action, J, is not varying much over the M turns. $K(n)$ is the correlation function of the σ_x variations over turn n:

$$K(n) = \sum_{m=0}^{M-1} \frac{\Delta\sigma_x(m)}{\sigma_{x0}} \frac{\Delta\sigma_x(m+n)}{\sigma_{x0}} \Big/ M. \tag{14}$$

The correlation function may be defined in terms of the power spectrum, $S_{\sigma_x}(\omega)$:

$$S_{\sigma_x}(\omega) = \sum_{n=1}^{M} K(n) \exp(i\omega n). \tag{15}$$

Using this expression in Eq. (13) and substituting into Eq. (12) we get

$$D(J) = 32\pi^3\Delta\nu_0^2 J^2 \exp\left(-\frac{J\beta}{\sigma_{x0}^2}\right) \times \sum_{k=0}^{\infty} (I_k - I_{k+2})^2 S_{\sigma_x}(\omega_k), \tag{16}$$

where $\omega_k = 2(k+1)2\pi[\nu_0 + \langle\Delta\nu(J)\rangle]$. The peak in the power spectra $S_{\sigma_x}(f)$ in Fig. 2(b) is near $4\pi\nu_0$ which corresponds to $k = 0$ in Eq. (16). Figure 1(d) shows the diffusion obtained from Eq. (16) for $k = 0$. The value of $S_{\sigma_x}(f)$ in Eq. (16) for a particular action J is obtained from measuring the frequency of the peak in the power spectra of the sample particle with that initial J [Fig. 2(a)]. Reasonable agreement between the δf computation and this analytic expression is found. If we used naive approximations for the correlation function such as the Lorentzian, we were unable to reproduce the exponential J dependence of $D(J)$ for $J > 0.5$.

In summary, through our extensive computation and theory we have discovered that the diffusion obtained from the self-consistent δf code is several orders of magnitude higher than that of the prediction from conventional tracking codes. The essence of the culprit of this enhanced diffusion is captured by the variation of the second moment of the beams $\langle x^2\rangle$ which is the result of beam-beam interaction induced collective variations of the beam distribution. This work was supported by U.S. DOE, Texas National Research Lab Commission, and SSCL.

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