An Outer Bound for Linear Multi-node Exact Repair Regenerating Codes

Marwen Zorgui, Zhiying Wang
Center for Pervasive Communications and Computing (CPCC)
University of California, Irvine, USA
{mzorgui,zhiying}@uci.edu

***Abstract*—In distributed storage, erasure codes provide fault-tolerance while reducing the storage overhead compared to replication. The network traffic cost during the repair of node failures, called repair bandwidth, is an important metric in code design. Regenerating codes are a class of erasure codes developed with the aim of reducing the repair bandwidth while maintaining a high level of fault-tolerance. They exhibit a tradeoff between the storage overhead per node and the repair bandwidth. However, a fundamental understanding of the storage-repair bandwidth tradeoff under exact repair is open in general. In this work, we consider the exact repair problem of multiple failures in a centralized way. Building upon techniques from the literature, we first provide an alternative proof of the functional repair bound. Then, we derive a new outer bound for linear centralized multi-node exact repair codes and illustrate its performance under various parameter settings. The derived outer bound shows that, in general, the centralized multi-node functional repair tradeoff is not achievable under linear exact repair.**

I. Introduction

In distributed storage systems (DSSs), a file is encoded into multiple fragments using an erasure code, and then stored across a number of nodes connected over a network. A serious challenge for DSSs is node failure, which may occur for several reasons. Indeed, as the system scales, the number of hardware failures scales with it. Moreover, the content of some nodes may be temporarily unavailable for several reasons, such as software updates, power outage, and network congestion. The repair problem is the problem of recovering the content of unavailable nodes, and it is a crucial task in DSSs. The total amount of information transferred through the network during a repair process is referred to as repair bandwidth. Regenerating codes [1], [2] are a class of novel erasure codes with efficient repair bandwidth.

The focus of this paper is on *centralized multi-node repair codes.* We assume all symbols belong to a finite field $\mathbb{F}$. A file of size $\mathcal{M}$ over $\mathbb{F}$ is stored in the system, such that each node stores α symbols in $\mathbb{F}$. The content of any k out of n nodes in the system is required to be sufficient to reconstruct the entire data. The repair of e nodes, where $1 \leq e \leq n-k$, is achieved by contacting any set of arbitrary d helpers such that $k \leq d \leq n-e$, and downloading $\beta \leq \alpha$ symbols. Furthermore, A code satisfying the centralized repair constraints is referred to as an $(\mathcal{M}, n, k, d, e, \alpha, \beta)$ regenerating code. We also say it is a code of the (n, k, d, e) system. If the repaired information is the same as the lost information, the code is *exact repair*. Otherwise, the code is *functional repair* as long as the reconstruction and repair requirements are maintained. We study outer bounds on centralized multi-node exact repair codes, which is open even for the single-node case in general.

Related work. Several constructions of regenerating codes repairing a single erasure have been proposed in the literature, e.g., [3]–[6]. Outer bounds for single-node exact repair include [7]–[12]. In particular, for linear $(n, k, k, 1)$ systems, the achievable schemes [5], [6] match the outer bound [9]–[11]. Following the lines of [11], the authors in [13] derived outer bounds for linear exact repair cooperative regenerating codes, where the communication among the replacement nodes also accounts for the repair bandwidth.

Constructions of centralized multi-node exact repair codes appear in, e.g., [6], [14]–[18]. Note that the functional repair bound represents an outer bound for the exact repair tradeoff. Multi-node functional repair regenerating codes satisfy [18]

$$\mathcal{M} \leq \min_{\mathbf{l}\in\mathcal{P}} \left(\sum_{i=1}^{g} \min(l_i\alpha, (d - \sum_{j=1}^{i-1} l_j)\beta) \right), \tag{1}$$

where

$$\mathcal{P} = \{\mathbf{l} = [l_1, \dots, l_g] : 1 \leq l_i \leq e, g \in \mathbb{N}, \sum_{i=1}^{g} l_i = k\}. \tag{2}$$

The vector $\mathbf{l}$ in (2) represents a scenario where the k contacted nodes are divided into g groups of sizes $l_i, i = 1, \dots, g$, respectively, such that the first l_1 nodes have been repaired simultaneously, the l_2 nodes repaired simultaneously, and so on [15], [18]. Moreover, [18] showed that most points on the functional repair tradeoff are not achievable under exact repair. Table I summarizes the relevant works, and the focus of this paper is on deriving outer bounds for exact-repair codes when $e > 1$.

Exact-repair codes	Achievable points	Outer bounds
$e = 1$	[3]–[6]	[1], [7]–[12]
$e > 1$	[6], [14]–[18]	[15], [18]

TABLE I: Exact-repair regenerating codes references.

Contributions of the paper. We provide an alternative proof of the functional repair bound. We then derive a new outer bound for linear multi-node exact repair regenerating codes (Theorem 1). We illustrate numerically the new bound, improving on the functional repair outer bound under various parameter settings. The techniques used in this paper follow along the lines of [11], [13]. The proof of Theorem 1 is given in the following sections.

Theorem 1. *Consider a linear exact repair regenerating code with parameters $(\mathcal{M}, n, k, d, e, \alpha, \beta)$, the file size $\mathcal{M}$ satisfies*

$$\mathcal{M} \leq \frac{s-1}{s+1}(d+e)\alpha + \frac{2}{s(s+1)} \sum_{h=1}^{g} \min(sl_h\alpha, hl_h\alpha, (d-k+\sum_{t=1}^{h} l_t)\beta), \tag{3}$$

where $\mathbf{l} = [l_1, \dots, l_g] \in \mathcal{P}$*, given by* (2)*, and* s *is an arbitrary integer with* $1 \leq s \leq g$*.*

Notation. For a non-negative integer n, define $[n] \triangleq \{1, \dots, n\}$. For integers $m \leq n$, define $[m, n] \triangleq \{m, m+1, \dots, n\}$. The transpose of a matrix M is denoted by M^t. The $(m \times m)$ identity matrix is denoted by I_m. Let M be a matrix consisting of mn submatrices as

$$M = \begin{bmatrix} M_{1,1} & \cdots & M_{1,n} \\ \vdots & \ddots & \vdots \\ M_{m,1} & \cdots & M_{m,n} \end{bmatrix}.$$

$S(M)$ denotes the column space of M. Consider sets $I \subseteq [m], J \subseteq [n]$, we denote by $M_{I,J}$ the matrix constructed by collecting submatrices whose indices i and j are in I and J, respectively. For $i \in [m], j \in [n]$, call $M_{i,[n]} = [M_{i,1} \dots M_{i,n}]$ the *thick row* indexed by i and $M_{[m],j} = \left[M_{1,j}^t \dots M_{m,j}^t\right]^t$ the *thick column* indexed by j.

II. Properties of linear exact repair codes

In this section, we define linear multi-node exact repair regenerating codes, and establish some of their properties.

An $(\mathcal{M}, n, k, d, e, \alpha, \beta)$ centralized repair regenerating code encodes a $(1 \times \mathcal{M})$ message vector $\mathbf{m}$ into a $(1 \times n\alpha)$ codeword vector $\mathbf{c}$. The first node stores the first α symbols of $\mathbf{c}$, the second node stores the next α symbols, and so on. Let $\mathbf{c}_i$ denote the vector of α symbols stored at node i, such that

$$\mathbf{c} = [\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n].$$

A multi-node exact repair regenerating code is *linear* if encoding, decoding and repair are linear. Thus, a linear exact repair code can be regarded as an $(n\alpha, \mathcal{M})$-linear code, such that there exists an $(\mathcal{M} \times n\alpha)$ generator matrix G and an $((n\alpha - \mathcal{M}) \times n\alpha)$ parity check matrix H which satisfy

$$\mathbf{c} = \mathbf{m}G, GH^t = \mathbf{0},$$
$$\text{rank}(G) = \mathcal{M}, \text{rank}(H) = n\alpha - \mathcal{M}.$$

In the following, we restrict $n = d + e$, as any (n, k, d, e) system contains a $(d + e, k, d, e)$ system. We now derive a key lemma that will be used in the proof of Theorem 1, establishing conditions that must be satisfied by matrix H.

Lemma 1. *Consider an* $(\mathcal{M}, n, k, d, e, \alpha, \beta)$*-linear centralized exact repair code where* $n = d+e$*. The matrix* H *satisfies the following conditions.*

1) *The rank of a matrix constructed by collecting* $n - k$ *arbitrary thick columns of* H *is full, i.e.,* $(n-k)\alpha$*.*
2) *For any index set* $R = \{i_1, \dots, i_e\} \subseteq [n]$*, where* $i_1 < \dots < i_e$ *and* $|R| = e$*, there exists an* $(e\alpha \times n\alpha)$ *matrix*

$$H_R = \begin{bmatrix} A_{i_1,1}^R & \cdots & A_{1,n}^R \\ \vdots & \ddots & \vdots \\ A_{i_e,1}^R & \cdots & A_{e,n}^R \end{bmatrix},$$

where $A_{i,j}^R$ *is of size* $(\alpha \times \alpha)$*, such that*

a) $A_{R,R}^R = I_{e\alpha}$*,*
b) *for* $j \in D \triangleq [n] \backslash R$*, we have rank*$(A_{R,j}^R) \leq \beta$*,*
c) $S(H_R^t) \subseteq S(H^t)$*.*

Proof. The proof of 1) is identical to [13, Proof of Lemma 1] and is omitted. It follows from the data reconstruction property. We now prove 2).

Consider a repair process where the e nodes in R are centrally repaired with the d nodes in D. Node $j \in D$ sends a $(1 \times \beta)$ vector $\mathbf{s}_{R,j}$, which is a linear combination of elements of $\mathbf{c}_j$. That is, there exists an $(\alpha \times \beta)$ matrix $\Phi_{R,j}$ such that

$$\mathbf{s}_{R,j} = \mathbf{c}_j \Phi_{R,j}, j \in D.$$

From the repair requirement, $\mathbf{c}_i$, $i \in R$, can be reconstructed by linearly combining $d\beta$ symbols of $\{\mathbf{s}_{R,j}, j \in D\}$:

$$\mathbf{c}_i = \sum_{j \in D} \mathbf{s}_{R,j} \Psi_{i,j}^R = \sum_{j \in D} \mathbf{c}_j \Phi_{R,j} \Psi_{i,j}^R, \tag{4}$$

where $\Psi_{i,j}^R$ are encoding matrices of size $(\beta \times \alpha)$. For $i \in R$, define

$$A_{i,j}^R = \begin{cases} I_\alpha, & \text{if } j = i, \\ -(\Phi_{R,j} \Psi_{i,j}^R)^t, & \text{if } j \in D, \\ \mathbf{0}_{\alpha \times \alpha}, & \text{if } j \in R \backslash \{i\}. \end{cases} \tag{5}$$

Then, 2a) is proved. Moreover, for $j \in D$,

$$\begin{aligned} \left[(A_{i_1,j}^R)^t \quad \cdots \quad (A_{i_e,j}^R)^t\right] &= \left[-\Phi_{R,j}\Psi_{i_1,j}^R \quad \cdots \quad -\Phi_{R,j}\Psi_{i_e,j}^R\right] \\ &= \Phi_{R,j}\left[-\Psi_{i_1,j}^R \quad \cdots \quad -\Psi_{i_e,j}^R\right]. \end{aligned}$$

Thus,

$$\begin{aligned} \text{rank}(A_{R,j}^R) &= \text{rank}(\Phi_{R,j}\left[-\Psi_{i_1,j}^R \quad \cdots \quad -\Psi_{i_e,j}^R\right]) \\ &\leq \text{rank}(\Phi_{R,j}) \leq \beta. \end{aligned}$$

Finally, for every codeword $\mathbf{c}$, from (4) and (5) we have

$$\forall i \in R, \left[A_{i,1}^R \quad A_{i,2}^R \quad \cdots \quad A_{i,n}^R\right] \mathbf{c}^t = 0 \implies H_R \mathbf{c}^t = 0.$$

$S(H_R^t)$ is orthogonal to $S(G^t)$ and hence 2c) is proved. □

III. Outer bound for linear exact repair codes

In this section, we give an alternative proof of the functional repair bound (1) and prove Theorem 1 via the following steps:

1) Choose an arbitrary vector $\mathbf{l} \in \mathcal{P}$, defined in (2).
2) Using Lemma 1, construct an $(n\alpha \times n\alpha)$ repair matrix, $H_{repair,\mathbf{l}}$.
3) Find a lower bound for rank$(H_{repair,\mathbf{l}})$, hence a lower bound for rank(H).
4) An upper bound for $\mathcal{M}$ is obtained using the relation $\mathcal{M} = n\alpha - \text{rank}(H)$ and the lower bound for rank(H).

A. Construction of $H_{repair,\mathbf{l}}$

Consider a vector $\mathbf{l} \in \mathcal{P}$, and let $l_0 = n - k$. Then, $\sum_{i=0}^{g} l_i = n = d + e$. Consider the sets

$$R_h' = [\sum_{t=0}^{h-1} l_t + 1, \sum_{t=0}^{h} l_t], R_h = R_h' \cup N_h,$$

where $|R_h'| = l_h$, and N_h is chosen arbitrarily to satisfy $N_h \subseteq [\sum_{t=0}^{h-1} l_t], |N_h| = e - l_h$. For each set R_h, which is of size e, by Lemma 1, there exists a matrix H_{R_h} of size $(e\alpha \times n\alpha)$. From this matrix, we collect the last l_h thick rows in a matrix

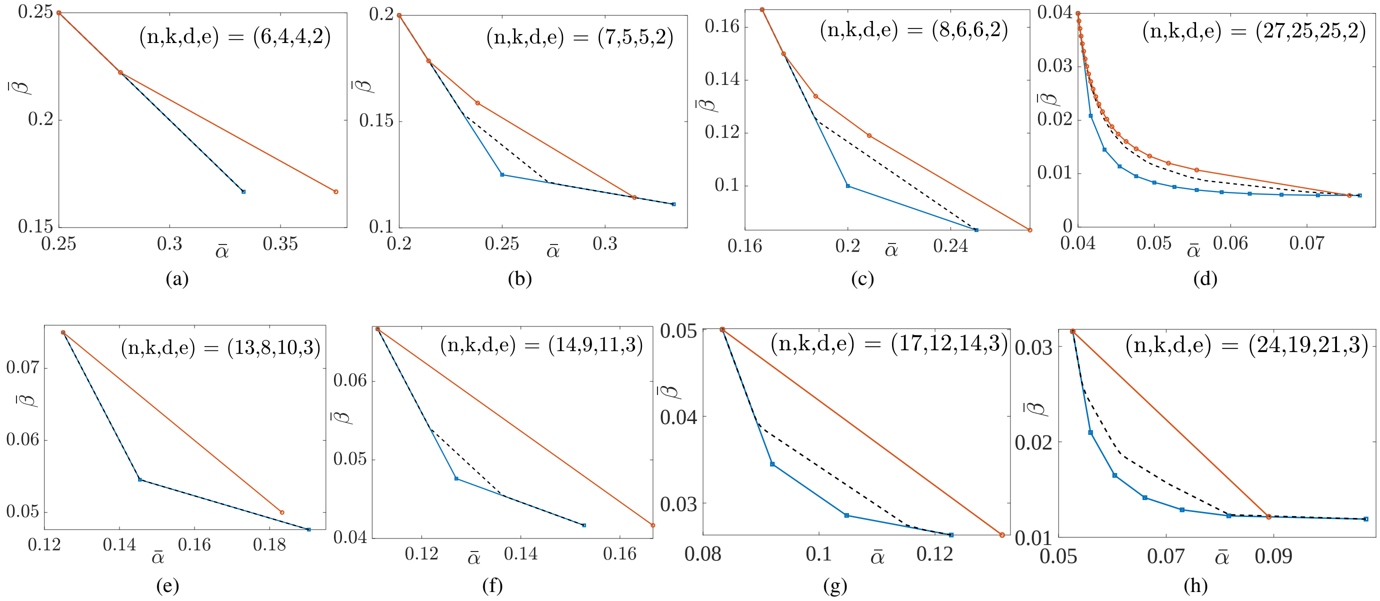

Fig. 1: Exact repair bound of Theorem 1 (black dashed curves) vs. functional repair bound (blue solid curves with squares) vs. achievable points from [17] (orange solid curves with circles), in terms of $\bar{\alpha} = \frac{\alpha}{\mathcal{M}}$ and $\bar{\beta} = \frac{\beta}{\mathcal{M}}$.

$A^{R_h}_{R'_h,[n]}$ of size $(l_h\alpha \times n\alpha)$. Note that in $A^{R_h}_{R'_h,[\sum_{t=0}^{h-1} l_t]}$, the set of thick columns that are zero correspond to N_h. The set of remaining thick columns are given by

$$L_h = [\sum_{t=0}^{h-1} l_t]\backslash N_h. \tag{6}$$

Let $\tilde{H}$ be constructed by taking the first l_0 thick columns of H and $\tilde{H}^\dagger$ its left-inverse of size $(l_0\alpha \times (n\alpha - \mathcal{M}))$. We construct $H_{repair,1}$ as follows:

$$H^t_{repair,1} = \begin{bmatrix} (\tilde{H}^\dagger H)^t & (A^{R_1}_{R'_1,[n]})^t & \cdots & (A^{R_g}_{R'_g,[n]})^t \end{bmatrix}.$$

By Condition 2c) of Lemma 1, $S(H^t_{repair,1}) \subseteq S(H^t)$. The thick rows of $H_{repair,1}$ have $l_0\alpha, l_1\alpha, \ldots, l_h\alpha$ rows, respectively. By grouping the columns by the same pattern, $H_{repair,1}$ consists of $(g+1)^2$ submatrices as

$$H_{repair,1} = \begin{bmatrix} B_{0,0} & \cdots & B_{0,g} \\ \vdots & \ddots & \vdots \\ B_{g,0} & \cdots & B_{g,g} \end{bmatrix},$$

where $B_{h,t}$ is of size $(l_h\alpha \times l_t\alpha)$ and

$$\begin{bmatrix} B_{0,0} & \cdots & B_{0,g} \end{bmatrix} = \tilde{H}^\dagger H,$$
$$B_{h,t} = A^{R_h}_{R'_h,R'_t}, \text{ for } 1 \le h \le g, 0 \le t \le g.$$

Moreover, $B_{h,h}$ is the h-th diagonal submatrix of size $(l_h\alpha \times l_h\alpha)$. It can be immediately checked that $B_{0,0} = I_{(n-k)\alpha}$, and from Condition 2a) in Lemma 1, it follows that $B_{h,h} = I_{l_h\alpha}$, for $1 \le h \le g$.

B. An alternative proof of the functional repair bound

Define $\delta_0 = \text{rank}(B_{[0,g],0})$ and

$$\delta_h = \text{rank}(B_{[0,g],[0,h]}) - \text{rank}(B_{[0,g],[0,h-1]}), \text{ for } 1 \le h \le g.$$

Then, δ_h represents the increment of rank after the h-th thick column is added. Note that $\delta_0 = (n-k)\alpha$. For the h-th thick row, $1 \le h \le g$, we have

$$\begin{aligned} \delta_h &\ge \text{rank}(B_{h,[0,h]}) - \text{rank}(B_{h,[0,h-1]}) \\ &\ge \text{rank}(B_{h,h}) - \text{rank}(B_{h,[0,h-1]}) \\ &= l_h\alpha - \text{rank}(A^{R_h}_{R'_h,[\sum_{t=0}^{h-1} l_t]}). \end{aligned} \tag{7}$$

By the definition of L_h in (6), we have $\text{rank}(A^{R_h}_{R'_h,[\sum_{t=0}^{h-1} l_t]}) = \text{rank}(A^{R_h}_{R'_h,L_h})$. By Condition 2b) in Lemma 1, we have

$$\text{rank}(A^{R_h}_{R'_h,j}) \le \text{rank}(A^{R_h}_{R_h,j}) \le \beta, \forall j \in [n]\backslash R_h.$$

Note that $L_h \subseteq [n]\backslash R_h$, and $|L_h| = \sum_{t=0}^{h-1} l_t - N_h = \sum_{t=0}^{h} l_t - e = d - k + \sum_{t=1}^{h} l_t$. Therefore, we have

$$\text{rank}(A^{R_h}_{R'_h,L_h}) \le \sum_{j\in L_h} \text{rank}(A^{R_h}_{R'_h,j}) \le \beta|L_h| = \beta(d-k+\sum_{t=1}^{h} l_t). \tag{8}$$

Thus, combining (7) and (8), we have

$$\begin{aligned} \text{rank}(H_{repair,1}) &= \sum_{h=0}^{g} \delta_h = (n-k)\alpha + \sum_{h=1}^{g} \delta_h \\ &\ge (n-k)\alpha + \sum_{h=1}^{g} [l_h\alpha - (d-k+\sum_{t=1}^{h} l_t)]^+ \\ &= (n-k)\alpha + \sum_{h=1}^{g} [l_h\alpha - (d - \sum_{t=h+1}^{g} l_t)\beta]^+, \end{aligned}$$

as $\sum_{t=1}^{g} l_t = k$, where $[x]^+ \triangleq \max(0,x)$. Therefore, we have

$$\begin{aligned} \mathcal{M} = n\alpha - \text{rank}(H) &\le n\alpha - \text{rank}(H_{repair,1}) \\ &= \sum_{h=1}^{g} l_h\alpha - \sum_{h=1}^{g} [l_h\alpha - (d - \sum_{t=h+1}^{g} l_t)\beta]^+ \\ &= \sum_{h=1}^{g} \min(l_h\alpha, (d - \sum_{t=h+1}^{g} l_t)\beta), \end{aligned}$$

which is equivalent to (1).

C. Derivation of Theorem 1

We now obtain a different lower bound for $\mathrm{rank}(H_{repair,\mathbf{l}})$ using the following theorem.

Theorem 2 ([13]). *Consider a matrix M with n thick columns and n thick rows (i.e., M has n^2 submatrices, denoted $M_{i,j}, i, j \in [n]$). The number of columns (rows) in each thick column (thick row) does not need to be identical. If M satisfies the following conditions*

1) *For any $j \in [n]$, the thick column $M_{[n],j}$ has linearly independent columns,*
2) *$\forall j \in [n], rank(M_{j,j}) = rank(M_{[n],j})$,*

then, for every integer $s \geq 1$, $rank(M)$ is lower bounded by

$$\frac{s(s+1)}{2} rank(M) \geq \sum_{i=1}^{n} \max(0, (s-i+1) rank(M_{i,i}), s\ rank(M_{i,i}) - T_i), \quad (9)$$

where $T_i = \sum_{j=1}^{i-1} \ rank(M_{i,j})$ for $2 \leq i \leq n$, and $T_1 = 0$.

For a given vector $\mathbf{l}$, $H_{repair,\mathbf{l}}$ has $(g+1)^2$ submatrices. Moreover, the diagonal submatrices are identity matrices. Thus, $H_{repair,\mathbf{l}}$ satisfies the two conditions in Theorem 2. Consider an integer $s \geq 1$. Using (9), we obtain

$$\frac{s(s+1)}{2} \mathrm{rank}(H_{repair,\mathbf{l}}) \geq s(n-k)\alpha + \sum_{h=1}^{g} \max(0, (s-h) l_h \alpha, s l_h \alpha - T_h).$$

Recall L_h as defined in (6), then

$$\begin{aligned} T_h &= \sum_{t=0}^{h-1} \mathrm{rank}(B_{h,t}) = \sum_{t=0}^{h-1} \mathrm{rank}(A^{R_h}_{R'_h, R'_t}) \\ &\leq \sum_{j=0}^{\sum_{t=0}^{h-1} l_t} \mathrm{rank}(A^{R_h}_{R'_h, j}) = \sum_{j \in L_h} \mathrm{rank}(A^{R_h}_{R'_h, j}) \\ &\leq |L_h| \beta = (d-k+\sum_{t=1}^{h} l_t)\beta \triangleq \Delta_h. \end{aligned}$$

Thus, we write

$$\begin{aligned} \mathrm{rank}(H_{repair,\mathbf{l}}) \geq& \frac{2(n-k)\alpha}{s+1} \\ &+ \frac{2}{s(s+1)} \sum_{h=1}^{g} \max(0, (s-h) l_h \alpha, s l_h \alpha - \Delta_h) \\ \geq& \frac{2(n-k)\alpha}{s+1} + \frac{2}{s(s+1)} \sum_{h=1}^{g} s l_h \alpha \\ &+ \frac{2}{s(s+1)} \sum_{h=1}^{g} \max(-s l_h \alpha, -h l_h \alpha, -\Delta_h), \\ =& \frac{2n\alpha}{s+1} - \frac{2}{s(s+1)} \sum_{h=1}^{g} \min(s l_h \alpha, h l_h \alpha, \Delta_h). \end{aligned}$$

In terms of $\mathcal{M}$, we have $\mathcal{M} \leq n\alpha - rank(H_{repair,\mathbf{l}})$, which simplifies to (3) in Theorem 1. While (3) holds for $s \geq 1$, following [13, Remark 6], it is sufficient to consider $s \in [g]$.

Remark 1. *When $s = 1$, (3) coincides with the bound in (1). Theorem 1 is at least as tight as the functional repair bound. When $e = 1$,* (3) *coincides with the bounds in [11].*

IV. Evaluation of Theorem 1 and Discussion

We evaluate Theorem 1 under various parameter settings in Fig. 1. The bound improves upon the functional repair bound when d is close to k. For instance, for the $(k+e, k, k, e)$ system, we observe that the exact repair bound approaches the inner bound from [17] as k increases, for fixed e. We note however that Theorem 1 does not rule out the achievability of the minimum bandwidth point of the functional tradeoff, which is shown to be not achievable under linear exact repair in [18]. This letter is a step toward understanding the fundamental storage-bandwidth under linear exact repair. Developing matching outer and inner bounds is still open and represents an interesting research direction.

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